GRADE 12 APPLIED
UNIT C – FUNCTIONS
CLASS NOTES FRAME

INTRODUCTION

1. In Grade 10 Applied (and Grade 11 Essential) we studied lines. Each change in **x** had a proportional change in **y**. In **Grade 12 Applied** we will study more '***functions***' (relationships between sets of values) that are not just simple 'linear' relationships. In Grade 11 Applied you studied Quadratic Functions; functions with a 'squared' variable in the *general form* $\mathbf{y = ax^2 + bx + c}$.

Pre-requisites. In order to be readily successful in this unit you will want to be familiar with Integer Operations (negative numbers), Linear Functions, Grade 11 Applied Quadratic Functions, Graphing points on Scatter Plots, Exponents, and Algebra.

What is a function?

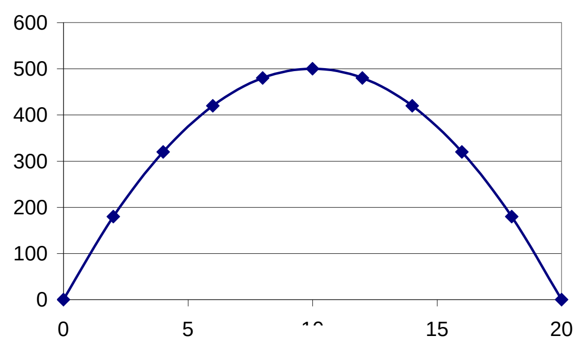
2. Go to the Appendix to these notes to review what a function is.

The next dozen pages are effectively a verbatim duplicate of Grade 11 Applied Quadratic Functions notes*.

GRAPHS AND REAL LIFE

3. The best way to see a relationship between sets of numbers is to look at a graph. What might the following graphs represent? Or as we say: what might the graphs 'model' in the real world?

a.



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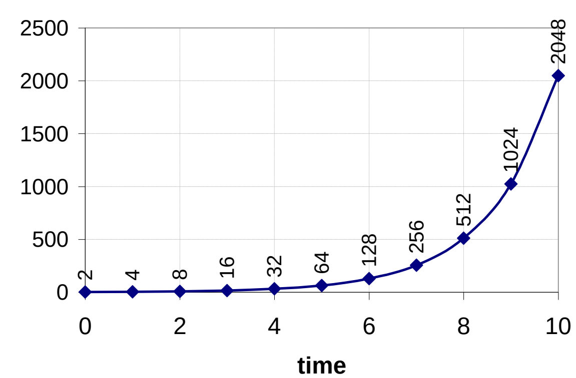

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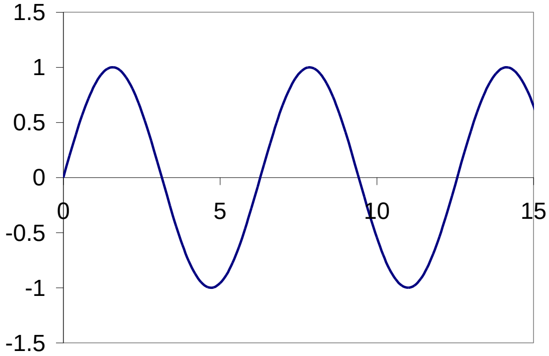

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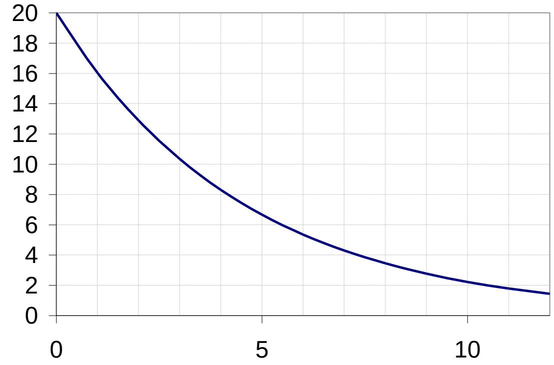
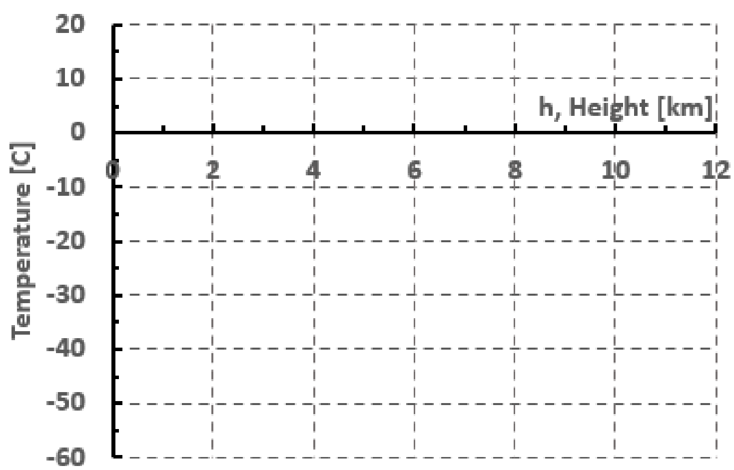


DEGREES OF FUNCTIONS (we will better examine what a function is shortly)

4. We had already studied **linear** functions. They were a function (relationship) where the ***y* = m*x* + b**. So, for example the **y** might be double the **x** and add **five**; **y = 2x + 5**. The ***x*** and ***y*** in the equation were just plain variables; no powers or square roots or reciprocals or anything fancy. A good example is that the temperature of the air is $\boldsymbol{T = -6.6 * h + 15}$ where temperature **T** is measured in degrees Celsius [°C] and , h, height above sea level is measured in units of km. A simpler linear relationship is that your age = mom's age – 20, for example.

Plot a graph of the Air Temperature as a function of height above sea level.

h [km]	**-6.6*h+15**	**T [°C]**
0		
1		
2		
3		
5		
10		



A rather simple linear relationship, as the height increases the temperature gets proportionately colder. Every increase in height by 1 km is a decrease in temperature of 6.6°C. $\frac{\Delta \boldsymbol{T}}{\Delta \boldsymbol{h}} = -6.6$

5. There are certainly other types of *functions*. The next most important one is the *quadratic function*. It is of '*degree 2*'. It often looks like this: $y = ax^2 + bx + c$. Where the a and b (coefficients) and c (the constant), can be found to represent real world situations that relate the x numbers to the y numbers. Notice it is a *polynomial* expression, a bunch of powers of *x* added together.

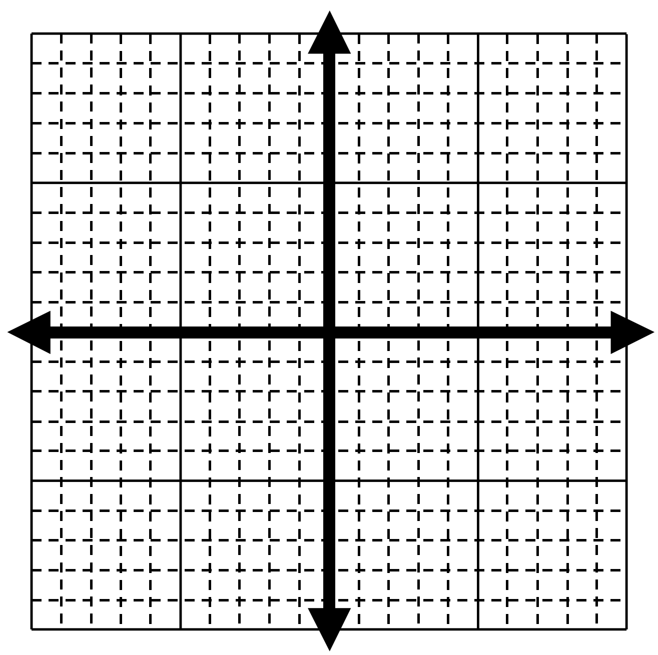
6. The degree of a polynomial equation is the highest exponent of any variable in a function. Thus, the equation of the function $\boldsymbol{y = ax^2 + bx + c}$ is of degree two. It is called a *quadratic* function since 'quad' means square in latin. Our former linear equation then was of degree one. $y = mx^1 + b$. We will have occasion soon to even study cubic equations of degree three! ($y = ax^3 + bx^2 + cx + d$)

CHARACTERISTICS OF THE QUADRATIC FUNCTION

8. **Manually calculate** (no calculator!) the data tables and **graph** the following two quadratic functions for given values of x in the associated table. You may find it necessary to make the scale of the *y*-axis different from the *x*-axis. (*Scaling* a graph)

a. $\boldsymbol{y = x^2 - 16}$

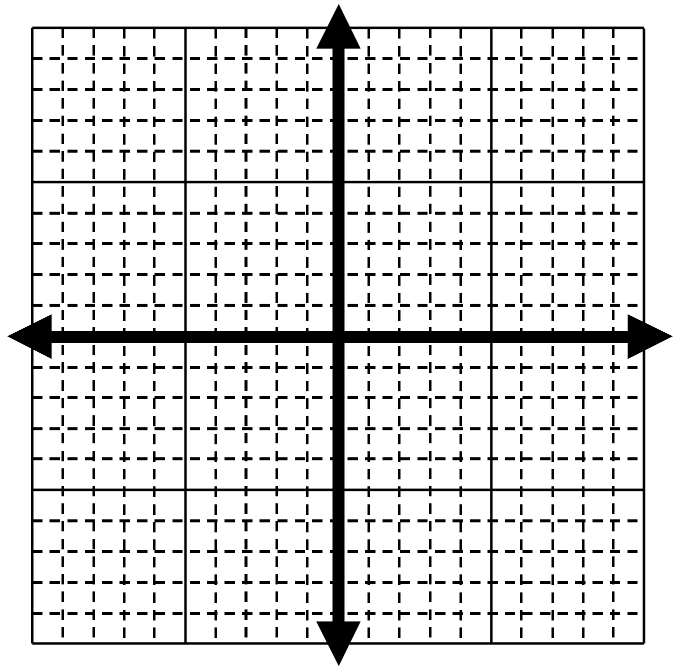
x	$x^2 - 16$	y
5	$(5)^2 - 16$	9
3		
2		
1		
0		
-1		
-3		
-5		



b. $y = -x^2 + 8x$

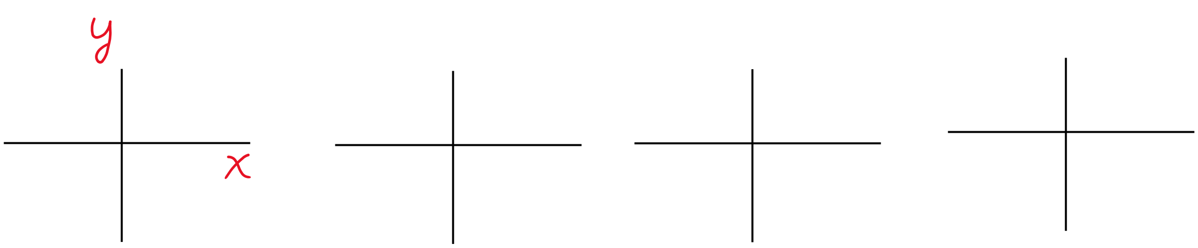
Scale the y-axis

x	$-x^2 + 8x$	y
-2		
-1		
0		
1		
3		
4		
5		
7		
9		



9. Notice that a *quadratic* equation has one 'bump'. It also has a point where the function reaches a **maximum** (*or* a **minimum**) 'vertex'. Also notice that a quadratic crosses (intercepts) the **y-axis** at exactly one place and one place only. It can cross (intercept) the **x-axis** at **two places or less**. Also notice that there is **symmetry** to the quadratic curve called a 'parabola', the left half of the parabola is the same as the right side, just a mirror image.

Let's Invent a few quadratics of our own and graph them with a graphing tool and sketch the result below. See if we can find a quadratic function where it doesn't cross the x-axis at all! See if you find a quadratic function where it only touches the x-axis in one place! Write their equations and sketch them below:



10. *Remarks on sketching*. Notice when we ask for a *sketch* I do not give accurate graph paper with precise tick marks. All we want in a sketch is a rough representation of what a graph looks like. The curve should have its significant points in the correct *quadrant* however. Had I given you more exact graph paper I would expect significant points on the graph to be accurate.

11. **Vertex**. The **point** (***x, y***) where a quadratic reaches a **maximum** or a **minimum** is called the **vertex**. Use the Desmos graphing tool. To graph the following quadratic functions. What is the vertex point for each of the two graphs below?

a. $y = x^2 - 16$ (____ , ____)

b. $y = -x^2 + 8x$ (____ , ____)

> *You will discover that when the **a** 'leading coefficient' is negative that the parabola opens down (cap) and when it is positive that the parabola opens up (cup)*

12. **Line of Symmetry**. The line of symmetry is just the **vertical line, x = constant,** that passes through the vertex. What are the lines of symmetry for the curves above?

a. $y = x^2 - 16$ **x =**__________

b. $y = -x^2 + 8x$ **x =** __________

> *The vertex and the line of symmetry can also be found by: '**–b/2a'**; but that is Pre-Calculus information!*

13. **Domain and Range**. Recall from Grade 10 Applied that **Domain** is all the values that the *input x* can have, and **Range** is all the subsequent values that the resulting *output y* can have. Also notice that with a quadratic, because it has a _______________ or else a ________________ that the range will *never* be all numbers. There will be some numbers that a quadratic function can never attain. Find the domain and range of the two functions above.

a. $\mathbf{f(x) = x^2 - 16}$) Domain: {______**< x <**________ }
Range: {______**< f(x) <**________**}**

b. $\boldsymbol{f(x) = -x^2 + 8x}$ Domain: {______**< x <**________ }
Range: {______**< f(x) <**________ }

14. **Intercepts**. The intercepts are where a function crosses a coordinate grid axis. The *y*-intercept is where the function crosses the *y*-axis (that is, where $x = 0$) and the *x* intercept(*s*) are where the function touches or crosses the *x*-axis (ie: where $y = 0$). For a quadratic there can be either 0, 1, or 2 places where a function touches the *x* axis. Use the DESMOS graphing tool to find the intercepts for the functions below:

a. $\mathbf{x^2 - 16}$
y-intercept: (_____, _____).
x-intercept(s): (_____, ______) and (______, _______)

b. $\boldsymbol{y = -x^2 + 8x}$
y-intercept: (_____, _____).
x-intercept(s): (_____, ______) and (______, _______)

APPLIED PRACTICE

15. The height, **h**, as a function of time, **t**, that you can throw a ball on the earth is given by:

$$\mathbf{h_{earth} = -5t^2 + 20t + 2}$$

where **h** is height in meters and **20** is the vertical speed at which *you* throw the ball (in meters per second). ***t*** is the time in seconds of the ball leaving your hand. The constant **2** is because the thrower releases the ball **2** meters from the ground. Don't worry about the physics of how we know this function.

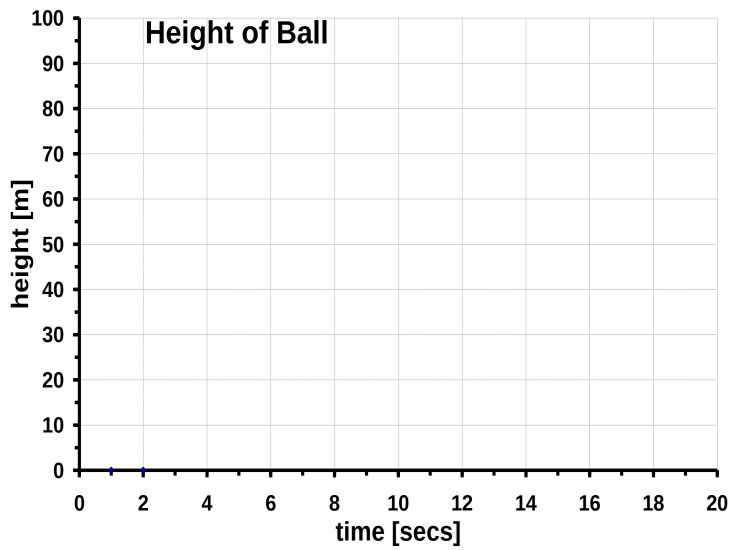
16. On the moon there is much less gravity, so the equation is:

$$\mathbf{h_{moon} = -1t^2 + 20t + 2}$$

(again, this is not physics class, do not worry about where I get the formula)

17. Calculate the data table. Manually graph the two functions (earth and moon) on the graph paper below. Use a Domain scale of 0 to 20 seconds along the x-axis (time). Use a Range scale of 0 to 100 meters on the y-axis (height).

Moon		Earth	
t	h_{moon}	t	h_{earth}
0		0	
5		1	
8		2	
10		3	
12		4	
15		5	
20			



[Workspace:]

a. where is the vertex of each function? Earth and moon?

Vertex Earth: (___,____)

Vertex Moon: (___,____)

b. what is the equation of the axis of symmetry *or* line of symmetry of each function?

Axis of Symmetry Earth:
x =_________.

Axis of Symmetry Moon:
x =_________.

c. what is the *y*-intercept of each function (ie: when time is zero)?

y-intercept Earth: or (0,____)

y-intercept Moon: or (0,____)

d. what are the (*approximate*) *x*-intercepts of each function? [Right most when ball hits the ground]

y-intercept Earth: (____,0)

y-intercept Moon: (____,0)

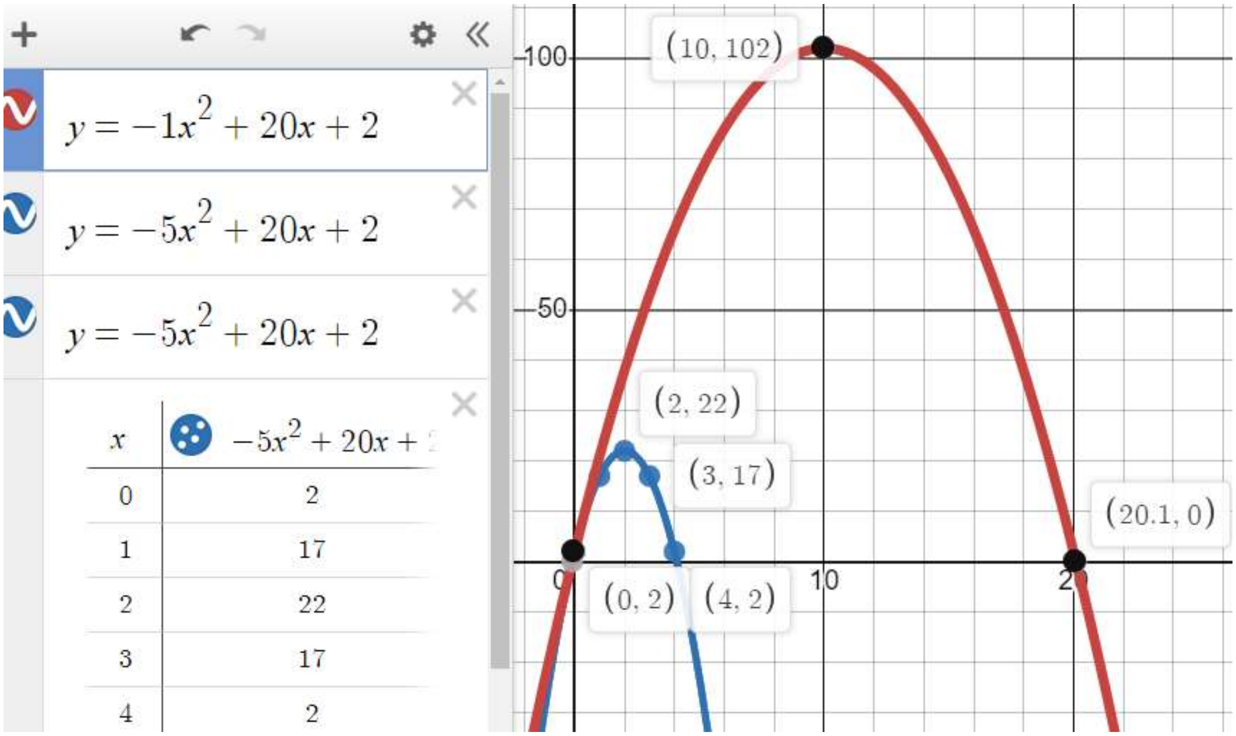
e. **Solving a quadratic Equation**! To solve means to determine when a function has a certain value!

Example: In how many years will you be 65? my age now + x = 65?

By inspecting the graph, approximately what time does the ball reach **20** meters height:

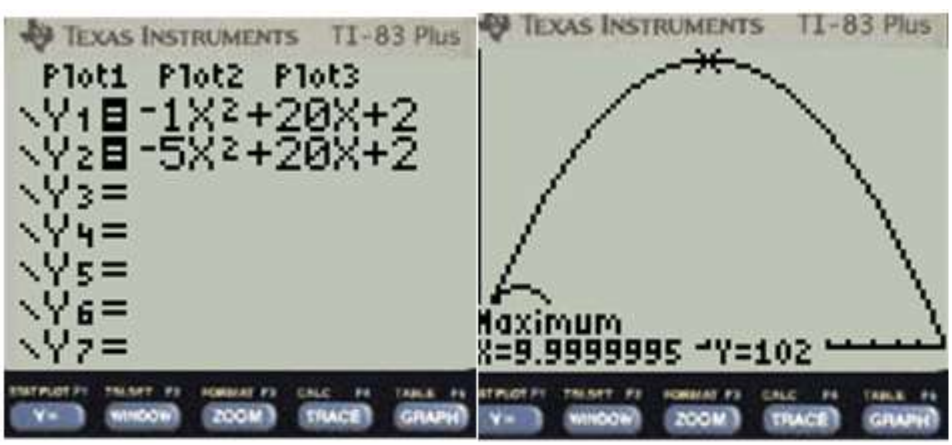
on the earth: _______________ ; on the moon: _______________

Now do all these graphical calculations using the DESMOS Graphing tool



Of course, there are tons of Apps and websites and calculators that will do the same type of graphing! Investigate those.

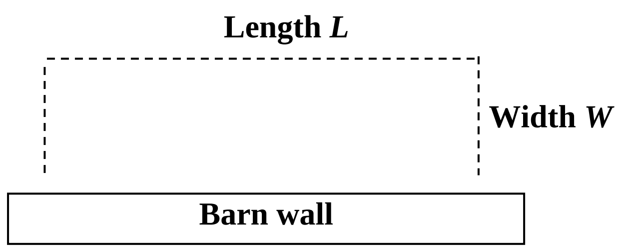
You are hopefully familiar with the Texas Instruments graphing calculators if you have done any proper math class in the last 25 years. Those graphing calculators, you may recall, are a bit more cumbersome and labour intensive to learn.



MAXIMIZING AND MINIMIZING STUFF

18. You have noticed that quadratic functions have a **maximum** or **minimum** value that they reach. Knowing when something reaches a peak, or bottoms out, or changes in a better direction is important in math and in life!

19. **Example**. Walter has an emu farm. He wants to fence off a **maximum** rectangular amount of area against his barn so that his *emus* have a *maximum* area to graze in. Walter has 30 meters of wire fence to make the enclosure.

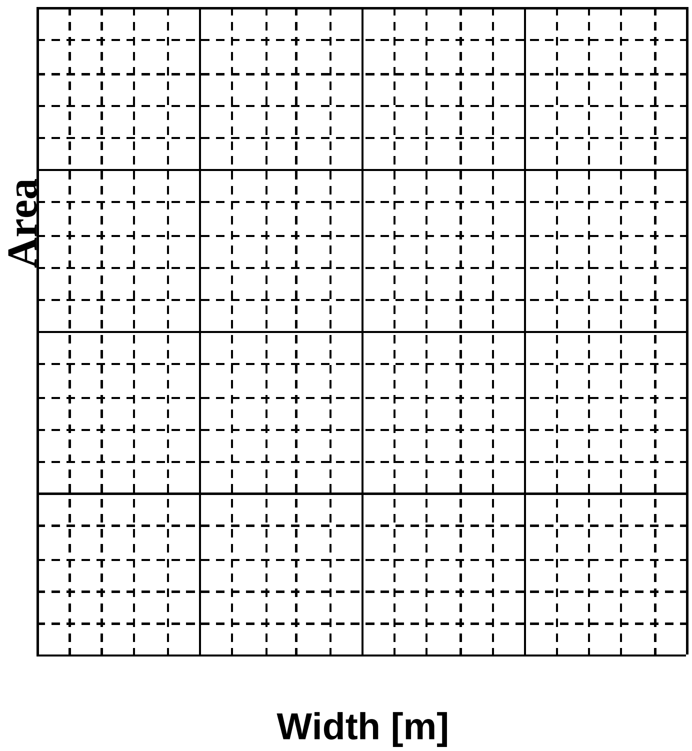


And of course he wants to minimize any cost, so he does not want to buy any more fence!

Maximizing profits and minimizing costs are a secret to business!

20. What is the *maximum area* he can fence off with his 30 metres of fence if one side is his barn? What dimensions of length and width will give him the maximum area? Complete the table below for a few select widths and lengths and manually graph it below.

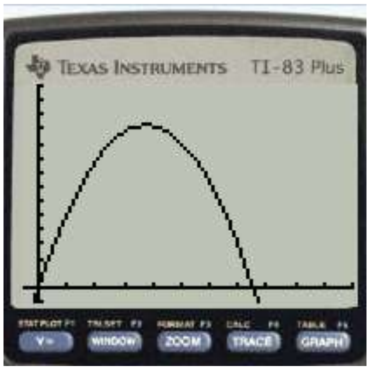
Hint: 2W + *L= 30* (so that L = 30 – 2W) and ∴Area = L*W



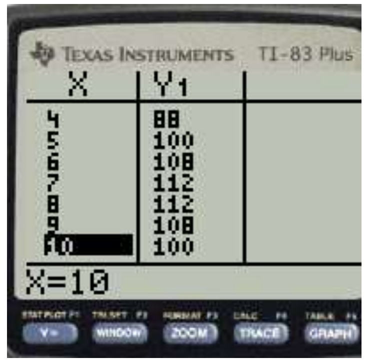
W [m]	L [m]	Area [m^2]

EMU farm Workspace:

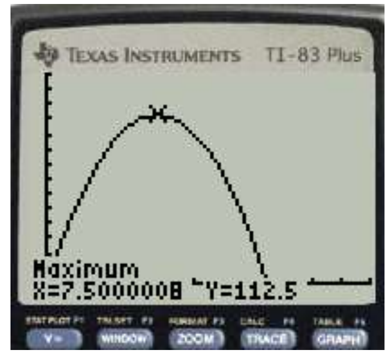
21. Check it all using a **TI-83** or any graphing tool and the formula **y = x*(30 – 2x)** where we are using **x** for the width now. Notice that **y = -2x² + 30x** is the more familiar polynomial general form when you multiply (**F.O.I.L.**) it out. Your TI-83 should have given results similar to these pictures below:



The Graph
For Emu pasture

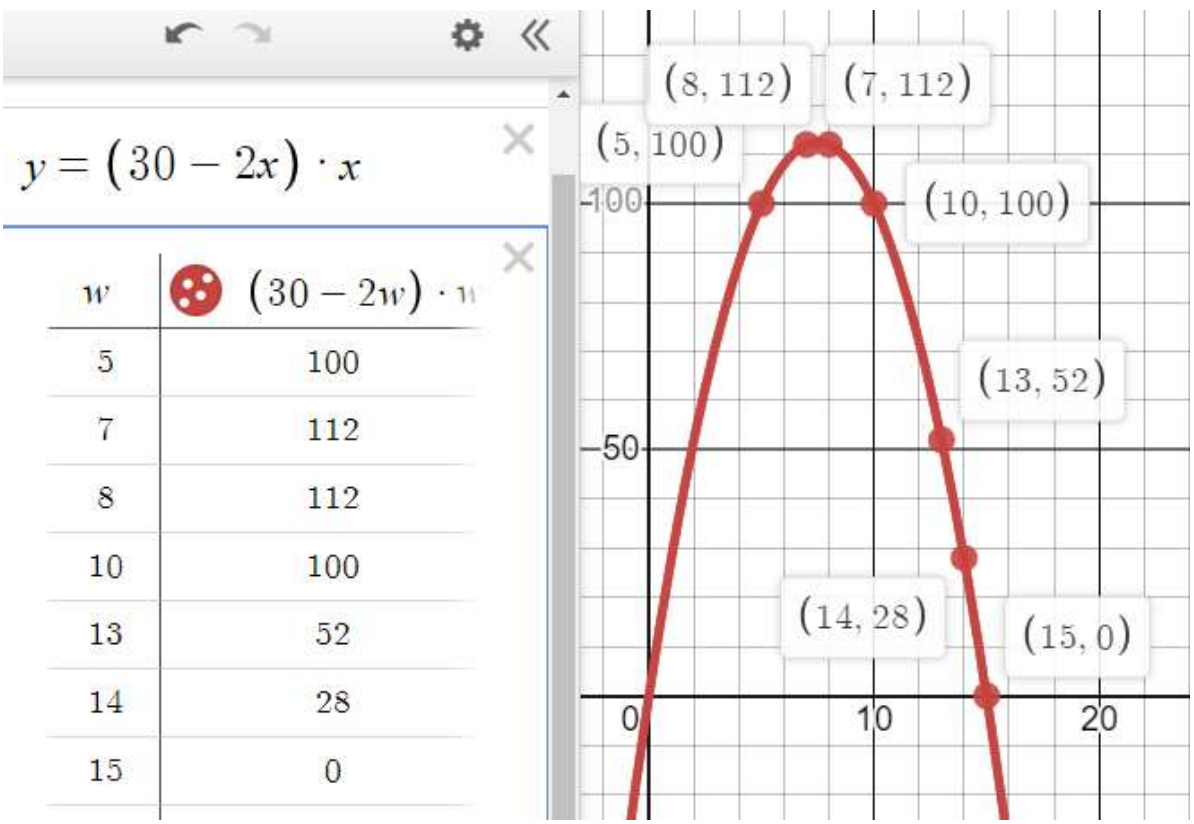


The Table



The Vertex (Maximum)

Here is the emu farm solution using the DESMOS graphing tool:



APPLIED EXAMPLE OF QUADRATIC– BRAKING DISTANCE!

30. The braking distance of car, once the brakes are applied, is a function of slipperiness of the road *and* of speed. In fact, the stopping (skidding) distance (not including reaction time) actually depends on the **square** of the speed! (*Really*! **Double** your speed: **quadruple** your stopping distance! Or quadruple the damage to your head when you hit a telephone pole). The formula for the braking distance then is like this:

$$D = .01 * K * v^2$$

In this formula, **D** is distance to stop in meters. The initial speed when you start braking, ***v***, is in km/hour. The **K** is an extra *coefficient* to allow for slipperiness and friction with the road. When **K** is **1** the road is **normal and dry**. When **K** is more than 1 **(K>1)** the road is slippery. On ice, **K**, can be as high as **5**.

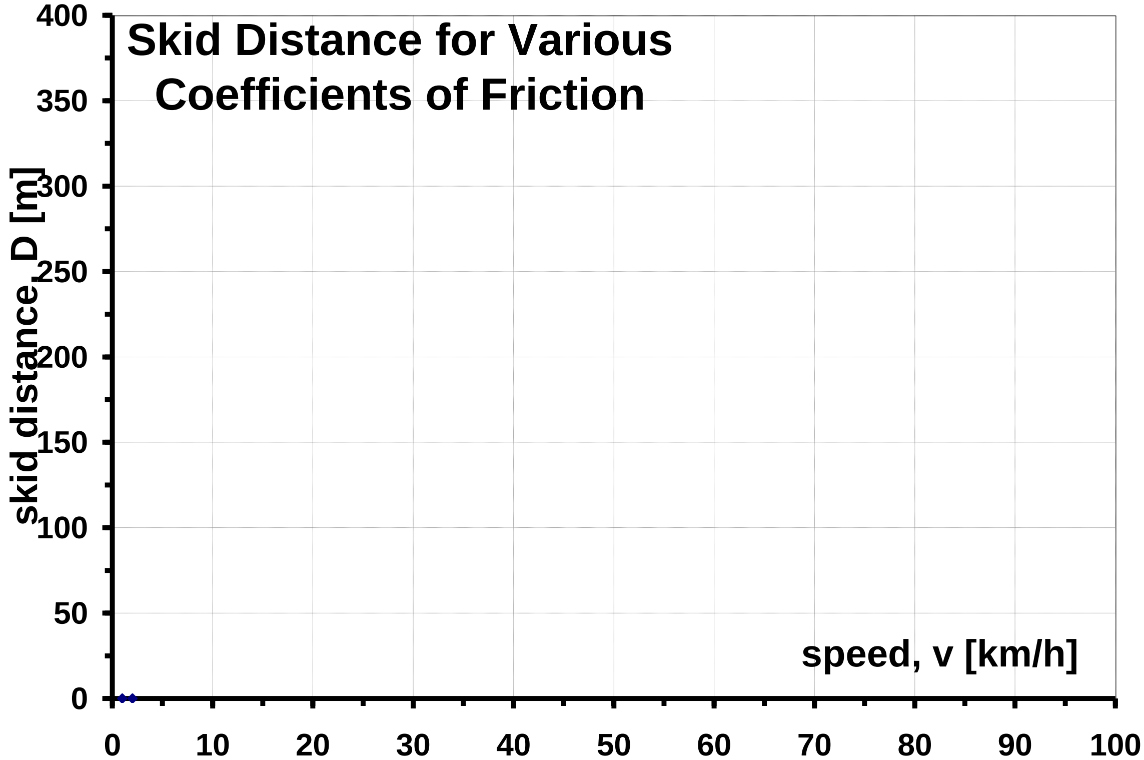
31. What distance does it take for the car to stop if initial speed, **v**, is **50** km/hr and the roads are dry, **K=1**. ?_________________ (hint: *evaluate*, plug in) $D = 0.01 \cdot 1 \cdot 50^2 = 25m$

32 What distance does it take for the car to stop if initial speed, **v**, is **100** km/hr and the roads are dry, **K = 1**. ?_________________

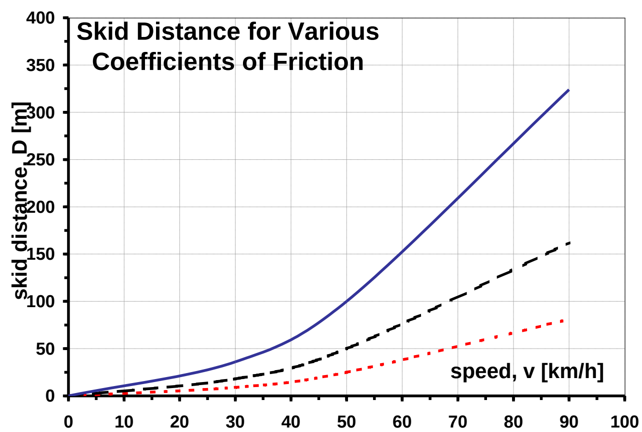
33. Complete the blanks in the stopping distance table:

Stopping Distance, *D*, Table for various Friction coefficients, K and velocities, *v*			
v	**K**	**D = .01*K*v^2**	**Value of D**
30	1	D =	
60	1	D =	
90	1	D =	
30	2	D =	
60	2	D =	
90	2	D =	
30	4	D =	
60	4	D =	
90	4	D =	
??	**4**	D =	**80 meters**
??	**1.75**	D =	**40 meters**

34. Now graph below the data from the table we calculated above!. (You will need to graph three separate parabolic curves, one for each given '**K**').



Your graph should resemble this:



Notice how by doubling your speed you quadruple your stopping distance!

Graph all three curves on the DESMOS Graphing Tool

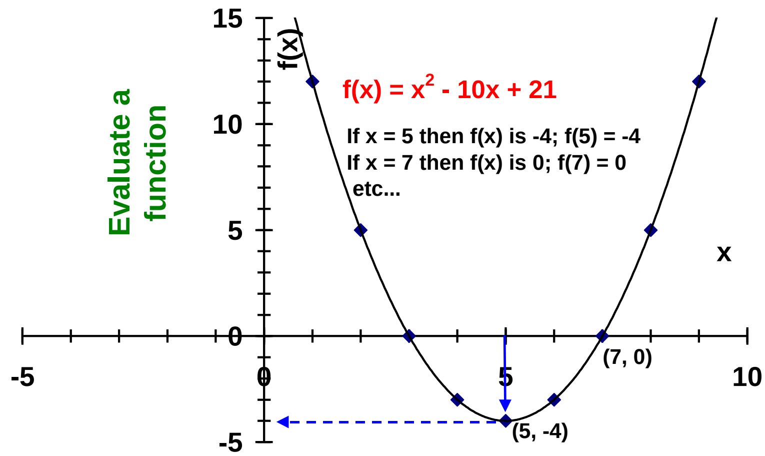
Check the table, check the graph.

Play with the DESMOS Graphing tool

SOLVE. How fast was someone going if their skid mark was 80 meters long on a dry day? ___________

SOLVING QUADRATIC EQUATIONS

55. So far we have '*evaluated*' some functions and '*graphed'* what they look like. To evaluate a function means to plug in different values of *x* to see what single value the function has for values for each x. And by graphing them we have seen where the functions have maximums, minimums, and where the function has a value of zero, etc.



56. **Solving.** Involves doing *evaluation backwards*. If you are told what the value of the function is **then what is the x** that gives you that **y-value**. **Example**: above, if you are told the function has a **value of 5** at some place(s) then what value(s) of x make that true?

x = ___________________________ (*notice there are two answers*)

57. **Solving using a graphing tool**. In Applied Math we use a graphing tool to solve functions. (In Pre-Calculus they learn how to solve many types of equations using only algebra). Any graphing tool will give you a *pretty close* answer for a solution to an equation but the TI-83 Graphing calculator has some very simple steps.

The DESMOS graphing Calculator is even easier!

58. To solve an **equation** such as $x^2 - 10x + 21 = 5$ (which means what value(*s*) of x makes the function have a value of **5**) with a **TI-83** graphing tool simply:

a. graph the function $x^2 - 10x + 21$ in **Y1=**

b. graph the function $y = 5$ in **Y2=** (A horizontal line in this simple case)

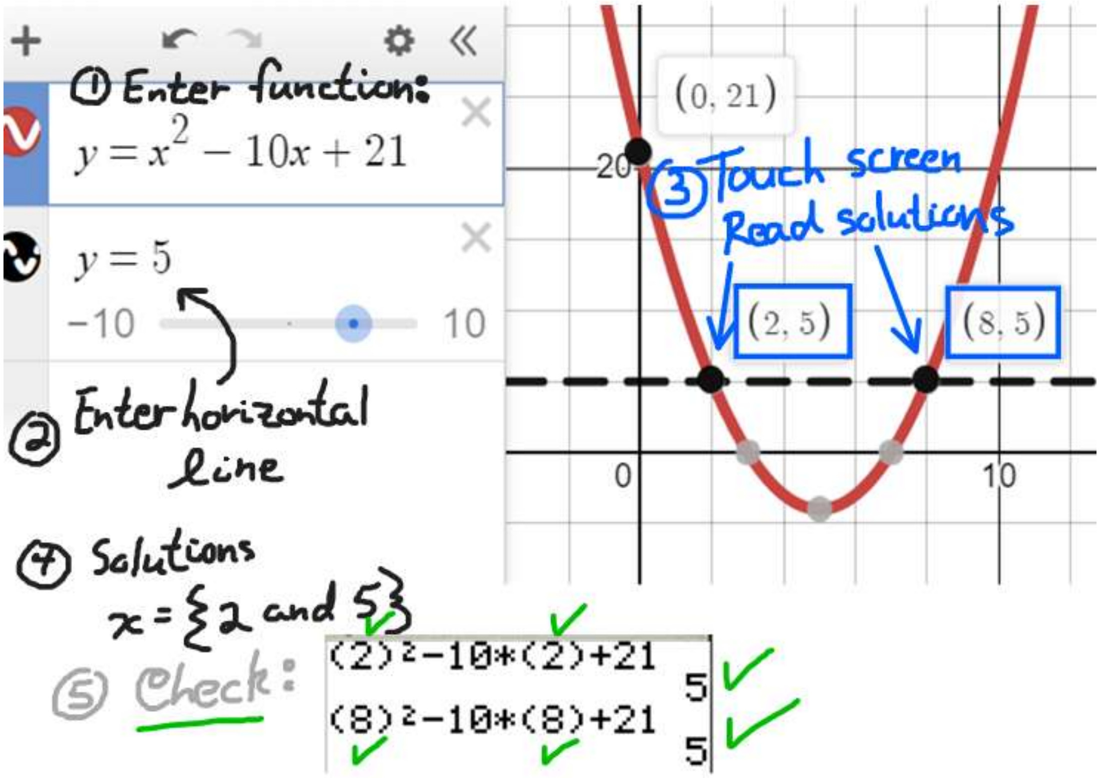
c. find the point(s) of intersection (where an ***x*** gives the value of 5) using the
2nd **TRACE** **5:INTERSECT** feature of the TI-83.

d. The screen will prompt you for which two curves you are interested in finding the intersection of. If you only have two functions entered in to **Y=** page then just hit **ENTER** twice.

e. Then move the '*spider*' close to the intersection you are looking for as a solution and hit **ENTER** again. The spider jumps exactly to where they cross and the place where they cross is displayed at the bottom of the screen as an ***x*** and a ***y***.

f. notice in this case there are two values of ***x*** that make the function
$x^2 - 10x + 21 = 5$. So you will need to do the intersection finding twice for the two points of intersection.

To solve with the DESMOS graphing tool is even easier.



59. Therefore the solution(*s*) to: $\mathbf{x^2 - 10x + 21 = 5}$ **is:**

x= ____________ **and x =** _______

YOU TRY

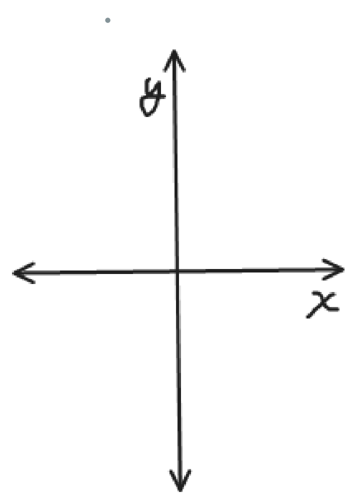
You try this one: solve for **x** given $x^2 - 5x + 6 = 8$ [to 2 decimal places]

The solution is

x = { −0.37 and +5.37 }

Indicate the solution(s) on your sketch.

Do a sketch of what your graphical solution looked like:



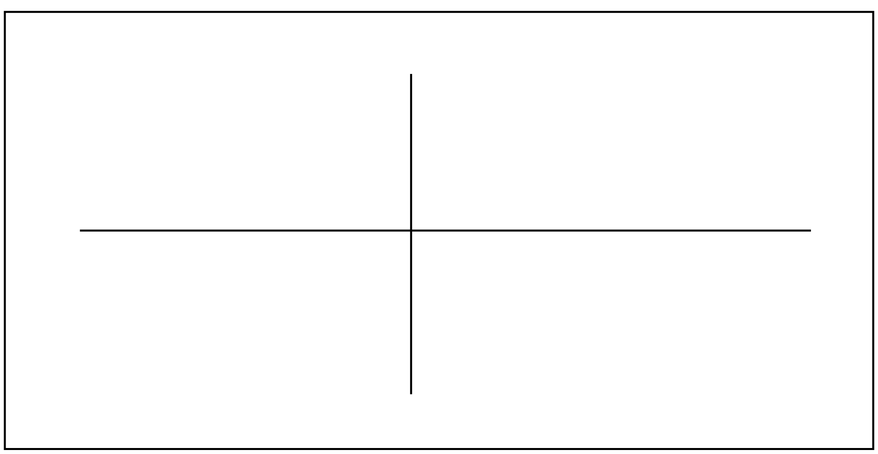
FYI. Graphing is easy. If you were doing **Pre-Calculus** to do any science or engineering or be a school teacher you would do some *basic* algebra like this:

$$x^2 - 5x + 6 = 8$$
$$x^2 - 5x + \frac{25}{4} + 6 = 8 + 25/4$$
$$(x - 5/2)^2 = 2 + 25/4$$
$$(x - 5/2)^2 = 33/4$$
$$x - 5/2 = \pm\sqrt{33/4}$$
$$x = 5/2 \pm \frac{\sqrt{33}}{2} = \boxed{\frac{5 \pm \sqrt{33}}{2}}$$
$$x \cong 5.37228 \text{ and } -0.37228\ldots$$

↑ GRADE 11 REVIEW ↑ IS COMPLETED

GRADE 12 CAN START NOW
CUBIC FUNCTIONS

60. A cubic function is one of *degree* three. Eg: $\mathbf{f(x) = x^3 - 4x}$. Graph this non-linear function with the value of the function on the y-axis with a graphing tool and **sketch** its graph below:



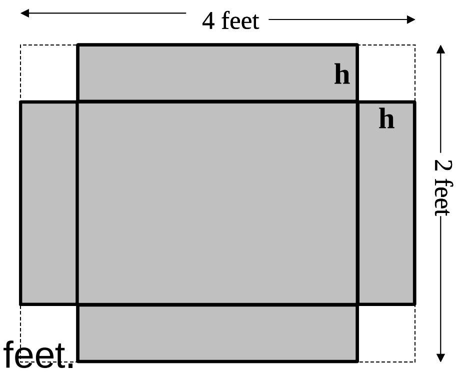
61. Notice that a cubic equation has two '*bumps'* in it (it often, but not always, has a 'local' maximum and a 'local' minimum too). These types of functions are common for finding volumes of Three-Dimensional figures. Recall from Grade 10 a few cubic functions we used such as:

$Vol_{sphere} = \frac{4}{3} radius^3$ or $Vol_{cube} = (edgelength)^3$

62. You want to make a cylindrical tank under your kitchen sink for your recycling. It can only be **40 cm** high. You want it to hold a **volume** of **0.5 cubic meters**. What radius must you make your tank? Recall the volume of a cylinder is given by:
$V = \pi r^2 h$ where ***h*** is the height and ***r*** is the radius. *Caution: you will need to convert the 40 cm into meters, since you cannot mix different measures of length in a formula*.

Ans: ________________ is the radius of the tank.

63. **Advanced Question**. The Nisga'a first Nation in BC loved your excellent Manitoba jingle dancers and granted you a piece of sacred cherry, a board 2 feet by 4 feet to make a 'box' for your jingle dresses. The box will have no lid. You recall that the volume of a box (or more correctly, a rectangular prism) is given by the formula Volume = Length * Width * Height.



a. draw out the '**net**' of the box→.

b. write the formula in terms of height.

Vol = **(4 – 2h) * (2 – 2h) * h**
Where h is in units of feet. Vol is cubic feet.

c. Graph the function on a Graphing tool to answer the following questions.

d. what is the volume of your box if you make the box 8 inches high? (careful! Watch units!!)

e. what is the volume of the box if you make box 12 inches high?

f. what is the volume of the box if you make it 6 inches high?

g. what height in inches (to the nearest 1/8th in) should you make the box to maximize its volume?

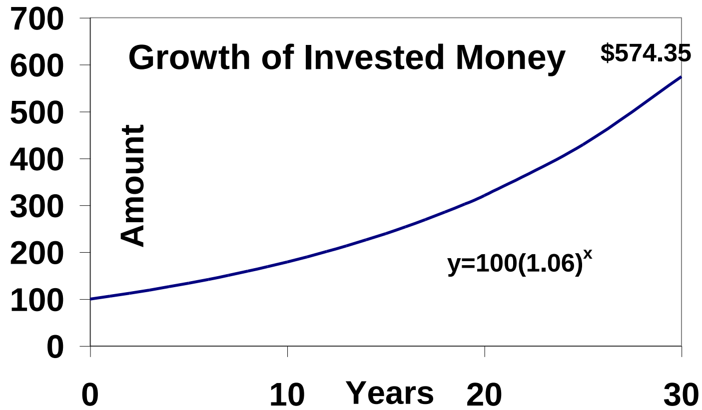
h. What are the value of all three dimensions (length, width, height) of the 'box' to maximize its volume?

i. what height would you make the box to give it a volume of exactly 1.5 cubic feet?

EXPONENTIAL FUNCTIONS

65. A very important function that you experience *every day* is the **exponential function**. It is called exponential because the **variable** is an **exponent**. A basic exponential function has the form: **$f(x) = ab^x$.** Exponential functions are ***not*** polynomial functions.

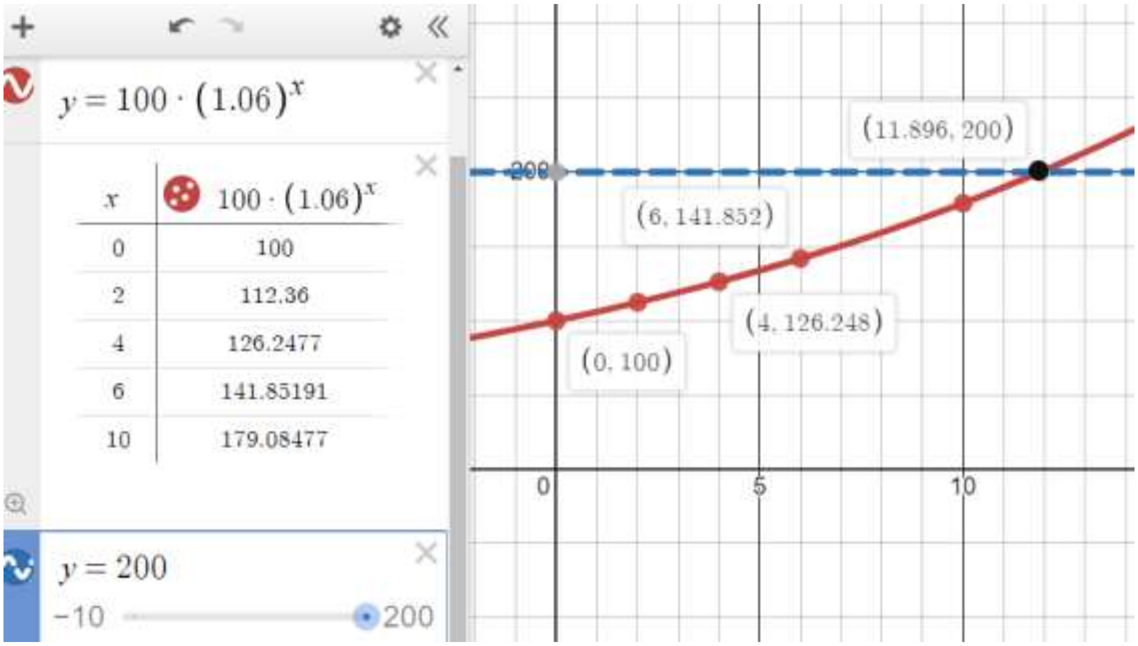
65. **An example**: The value of **100** dollars left to grow in a bank account at **6%** annual interest compounded annually can be given by **Value = $100\,(1.06)^t$**. Where ***t*** is the time in years.



a. How much money do you have in your account after 10 years? ________

b. When does your money double to $200.00? _____________

Try using the DESMOS Graphing tool for the exponential growth of that investment:



66. Mould and germs and diseases also grow exponentially, and the population of the earth grows exponentially (until they run out of food and resources of course, there is a different function for that though!).

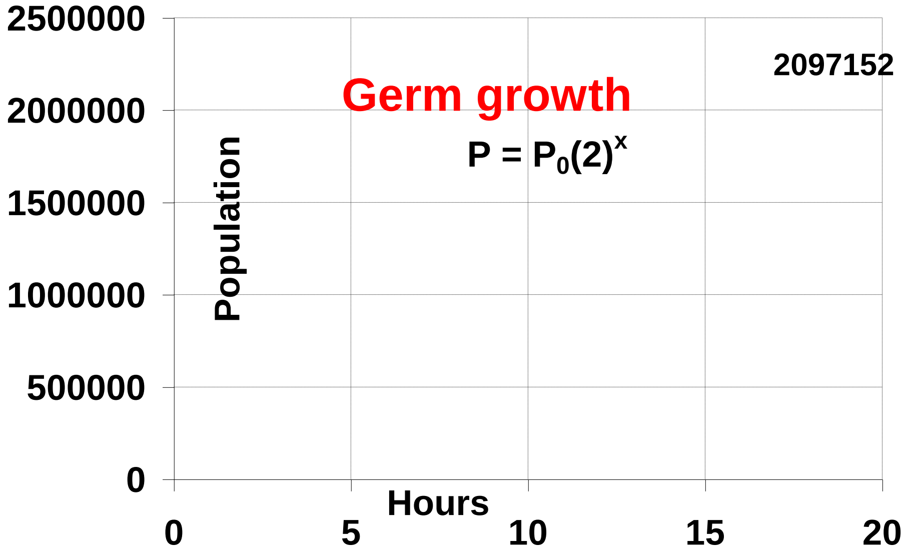
67. **Germ Growth Example**. Those nasty germs and viruses that you hate do grow *exponentially*. Say someone leaves two germs on a doorknob. Germs reproduce just like all living things; and say these germs double themselves every hour! Make a table of their growth:

Hours	0	1	2	3	4	5	6	10	20
# Germs	2	4	8	16	32				

Notice how our elapsed time above starts at time = 0, people often forget to start counting time at zero for some reason! Time of zero is 'now'!

68. Graph the Germ Growth on the next graph given that the equation is given by $\mathbf{P = P_0(2)^t}$; where **P** is the population (number of germs) at any particular time, $\mathbf{P_0}$ is the ***initial*** population (ie: when **t = 0**), and **t** is time in hours.

69. Drawing the exponential curve for the Germ Growth from above:

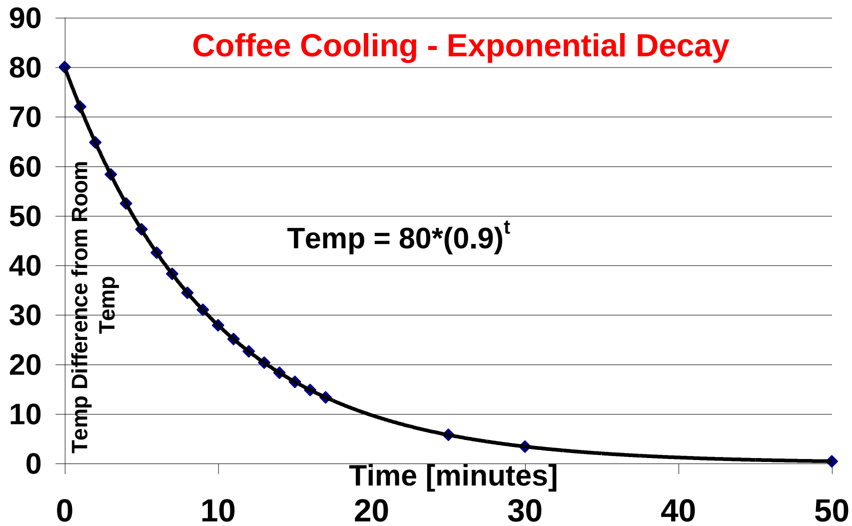


After 6.5 hours, how many germs on the door knob? ___________

At what time are there one million germs on the door knob? ______

70. **Exponential decay**. In the exponential form $\mathbf{y = ab^x}$, if the **base** is less than one but greater than zero (0 < b < 1), then the curve is said to ***decay***. Examples: Radioactivity, objects that cool, a sound dying out, etc are all said to decay ***exponentially*** as a ***function*** of time.

71. **Example Exponential Decay**. Have you ever noticed that your coffee seems to get cooler quicker when it is hot when you first get it? It cools more quickly in the beginning, but towards the end takes longer to cool off to room temperature. Let's say it is 0 degrees outside and you are walking to school with your coffee.



72. **Characteristics of Exponential equations**.

a. **Domain**: all real x. **(–∞ < x < ∞).** Consequently they have a y-intercept.

b. **Range**: notice the range *tries* to reach a **minimum** value at infinity (or negative infinity) in the x direction. The line that a curve gets *close too but never reaches* (eg: room temperature in the case of our coffee) is called the **Asymptote** line. Sketch in the asymptote in the coffee curve above.

c. **One-to-one**. Exponentials are *one-to-one functions*; they only have **one value** of **y** for each **x**, and each value of **y** has only one **x** that makes it, the y value is 'unique'. Functions like sine and quadratics and cubics nor any polynomial function in general are not one-to-one.

d. **Half Life**. In *exponential decay* situations things **decay,** not grow**,** by a certain percentage for each change in the **x,** usually time. In the case of our coffee the temperature dropped by **one half in 6.5 minutes** to **40C**, then in **another 6.5 minutes it dropped to half again** *from* **40C** *to* **20C**, then in **another 6.5 minutes (for a total of 19.5 minutes) it dropped half again** *from* **20C** *to* **10C**. The **time** it takes to **decay to one half** of a previous value is called a '**half-life**'. You will hear the term **half-life** a lot concerning radioactivity. The half-life of some radioactive substances can be thousands of years! So radioactive substances may be only **half as dangerous** as they were **every 10, 000 years** for example.

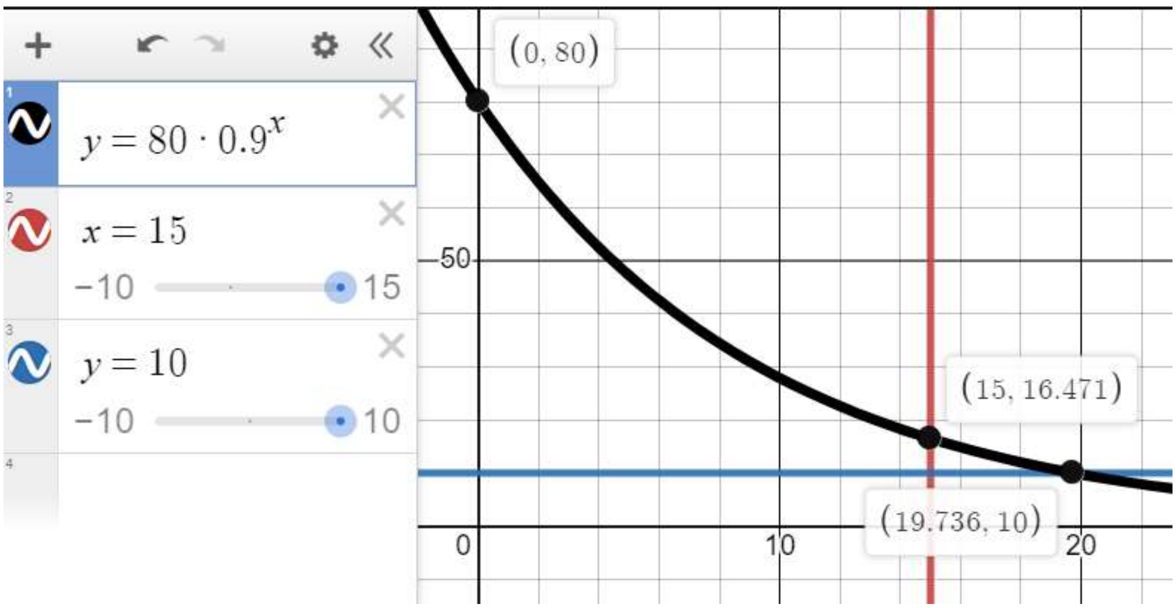
73. Calculate the following to 2 decimal places

a. What temperature is your coffee after 15 minutes? ________

b. When does your coffee cool to 10 degrees C? ________

c. Calculate the half life of your coffee's temperature using a graphing tool: _________

Now do the coffee cooling on a Desmos Graphing Tool!



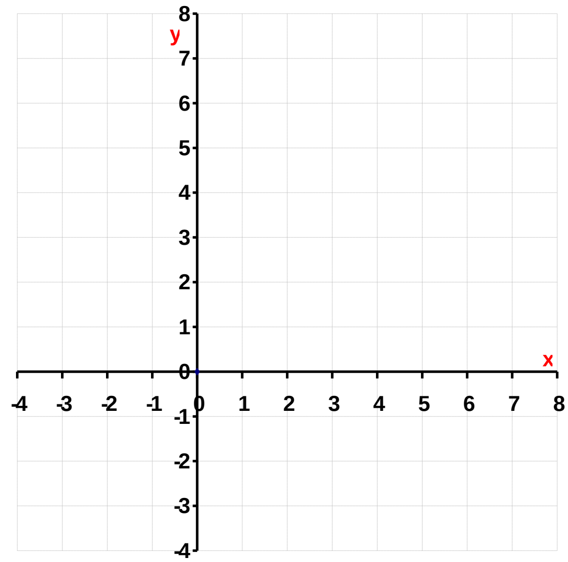
EASY!

GRAPHING AN EXPONENTIAL FUNCTION - BASICS

74. Manually calculate and graph the exponential function:

$f(x) = 2^x$

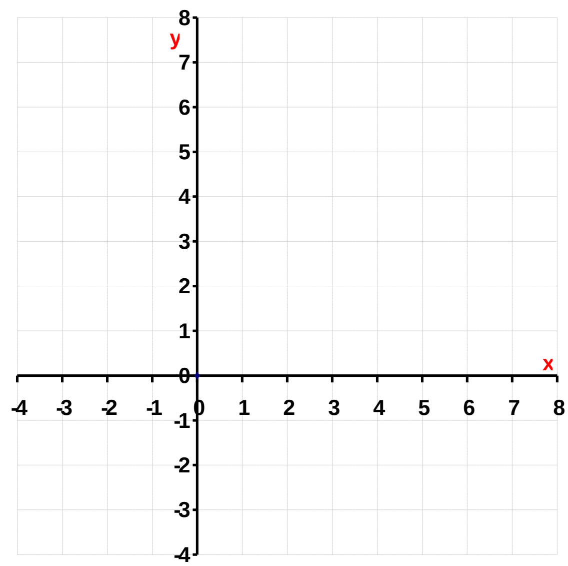
x	f(x)
⁻3	
⁻2	
⁻1	
0	
1	
2	
3	



75. Manually calculate and graph the exponential function:

$f(x) = \left(\frac{1}{2}\right)^x$

x	f(x)
⁻3	
⁻2	
⁻1	
0	
1	
2	
3	

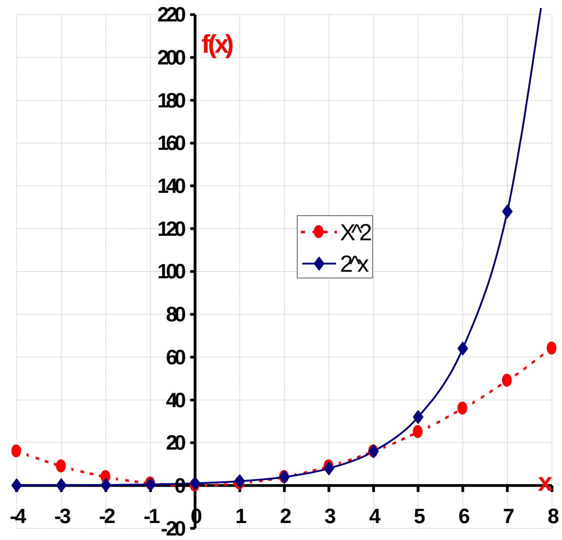


76. Make no mistake, an exponential is not the same as a quadratic! Notably, exponential functions 'take-off' eventually, even if they seem to start out slowly, and easily surpass polynomial functions in their y-values eventually. Complete the table to see how exponential functions grow rapidly after a slow start:

x	1	2	3	4	5	6	7	8
x^2	1	4	9	16	25			64
2^x	2	4	8					

78. The exponential function is '***one-to-one***'. The y-values are unique. When you solve for **y** there is only one answer. Unlike sine function or quadratic function.

79. The exponential function starts out slow but will readily surpass the power (polynomial) function such as $\mathbf{x^2}$.



LOGARITHMS

80. We use Logarithms to Solve exponential equations without graphing. Let's explore how we solve some of the basic equations we have learned so far.

81. Solve the following equations by using algebra (and check with a graphing tool)

a. **$2x + 1 = 7$** x =

b. **$2x + 1 = 3x - 4$** x =

c. **$x^2 = 9$** x =

tricky; two answers

d. $\sin \theta = 0.5$

tricky; two answers

e. $\sqrt{x-4} = 2$ x =

SOLVING EXPONENTIAL EQUATIONS USING LOGARITHMS

82. So how would you solve this using just algebra?

$$100*(1.05)^t = 200$$

(an exponential equation)(this is like asking when does my $100 become $200 at 5% compounding annually)

83. That is what your **LOG** and **Ln** buttons on your calculator do! They solve exponential equations; it allows you to undo an exponential equation of the form $y = ab^x$ to isolate the unknown variable **x** all by itself!

84. Try this fancy algebra: if we say we have a function y = 2^x and we want to know when the **y = 32**; in other words we want to solve the equation:

32 = 2^x

Enter this in your calculator: $\frac{\boldsymbol{log}(\mathbf{32})}{\boldsymbol{log}(\mathbf{2})}$; it had best say the answer is **5**. Of course you should check: is **2^5 = 32**?

85. Try this from a previous example above in which we graphed the solution:

The value of **100** dollars left to grow in a bank account at **6%** interest can be given by **Value = 100 $(1.06)^t$**. Where ***t*** is the time in years. When does you money double to **$200**.

86. Here are the algebra steps:

- **200 = 100$(1.06)^t$** *The equation to solve*
- $\frac{\mathbf{200}}{\mathbf{100}} = (\mathbf{1.06})^{\boldsymbol{t}}$ *Divide both sides by the co-efficient in front of the power to isolate the power.*
- *Simplify:* $2 = (1.06)^t$
- **log(2) = log$(1.06)^t$** *Perform the 'log' operation on both sides*
    - **log(2) = t * log(1.06)** *'Pop' the exponent down in front of the log*
- $\frac{log(2)}{log(1.06)} = t$ *Isolate the '**t**', divide by both sides by log (1.06)*
- *Use your calculator:* $\frac{log(2)}{log(1.06)} = 11.8957$ years.

Compare to your answer with the graphing calculator above.

87. **Ln Button**. By the way, the **Ln** button could have also been used above!

Solve the following exponential equations (show work!): Check with a graphing tool.

a. **$4,000 = 1,000 * 1.08^x$**	**Ans: x = 18.01**
b. **$20 = 90 * (0.93)^t$**	**Ans: t = 20.73**
c. **$515 = 2^x$** $+ 3$	**Ans: x = 9**

COMPARE LOGARITHM TO OTHER FUNCTIONS AND THEIR INVERSES

88. The logarithm function '*undoes*' the exponent function. You have been 'undoing' lots of functions ever since Grade 8. A function that *undoes* another function is called its ***inverse function***. Let's review a couple inverse functions and then the logarithm function.

Dividing undoes multiply

x→	2 * x
1	2
2	4
3	
	8
	12
x ÷ 2	←x

Complete the blanks

Square root undoes square

x→	x^2
2	4
3	9
10	
	16
	81
$\sqrt{x}$	←x

Complete the blanks
Going right is 'squaring'; going left is 'square rooting'

Logarithm *undoes* exponent

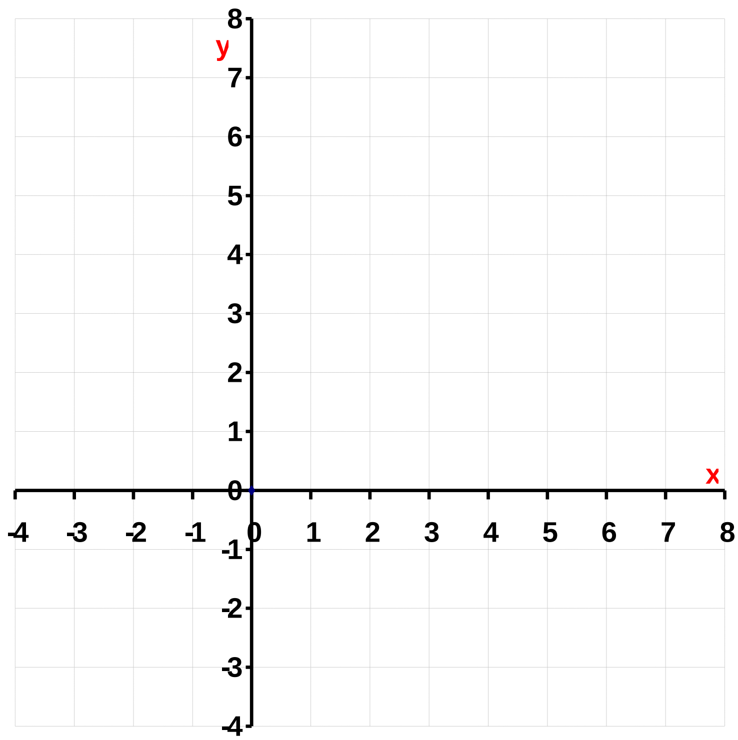
x→	10^x
0	1
1	10
2	100
	1000
	10,000
$\log_{10} x$	←x

Complete the blanks
A log undoes an exponent.
This is a base of 10; other bases are possible.

GRAPHING THE LOGARITHM FUNCTION

89. Since the Logarithm function just undoes the exponential function its graph should likely look somewhat similar?

90. Complete the table and graph and label the base 2 Exponential function *and* the base 2 Logarithm function.



x→	2^x
–3	
–2	1/4
–1	
0	1
	2
2	4
	8
$\log_2 x$	←x

Recall that $b^{-m} = \dfrac{1}{b^{+m}}$?

LOG$_{10}$ vs Ln = COMMON BASE 10 LOGARITHM vs NATURAL LOGARITHM

92. Your calculator has two LOG buttons. The LOG button itself is for base 10. Anytime you see just LOG it implies that it is for a base of 10 : **Log$_{10}$.**

A real mathematician or scientist would not use Base 10, they would use, ….

get ready for it, ……

‘**e**’.

WHAT IS THE NUMBER ‘e’?

94. To make things simple think of ‘**e**’ as being magic, sort of like π is magic. ‘**e**’ is most definitely a number; its approximate value is:
2.7182818284590452353602874713527…
and like π it is an *irrational number* that cannot be written as a fraction nor consequently as a decimal.

You will have to trust that ‘***e***’ really is magic, hang out with your teacher if you really want to know why.

95. **Ln is Log$_e$.** The **N**atural **L**ogarithm, **ln**, has ‘**e**’ as its base. Occasionally you will see **Ln** written as **Log$_e$.** For the purpose of this course the **Ln** button is sufficient for solving equations.

96. **Example - Solve Exponential Equation with ln**. The number of germs on a door knob grow *continuously* as a function of time in accordance with the exponential expression $\mathbf{N(t) = N_o * e^{t/4}}$. The key to knowing when the natural number ‘**e**’ is being used is when the problem talks about ‘*continuous* growth’. So in this example **N(t)** is the number of bacteria (whole numbers) as a function of time ***t***, $\mathbf{N_0}$ [pronounced ‘N naught’] is the initial number at time zero (naught in English), and **t** is time measured in hours.

97. So if we start with $\mathbf{N_0}$ = two germs then $\mathbf{N = 2e^{t/4}}$
Let’s find when we have a thousand germs on the door knob

$\mathbf{1{,}000 = 2e^{t/4}}$

$\mathbf{500 = e^{t/4}}$ — Divide both sides by 2 to isolate the power

$\mathbf{\ln(500) = \ln(e^{t/4})}$ — Perform Natural Log of both sides

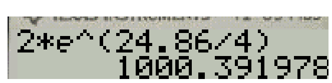
$\mathbf{\ln(500) = \frac{t}{4} \ln(e)}$ — Pop the exponent down in front of the ln

$$\frac{\ln(500)}{\ln(e)} = \frac{t}{4} \quad \therefore \quad \frac{4 * \ln(500)}{\ln(e)} = t$$

Isolate the t

t = 24.86 hours

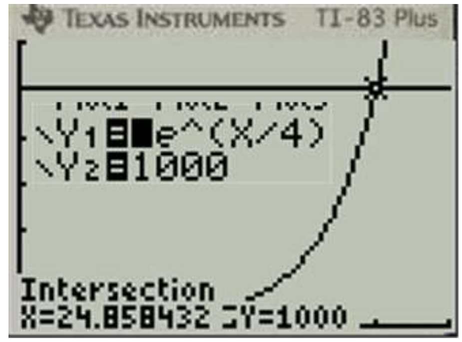
Solution: 24.86 hours to get 1,000 germs

Check answer!! $\mathbf{1{,}000 = 2e^{24.86/4}}$ **???** 

Yes!

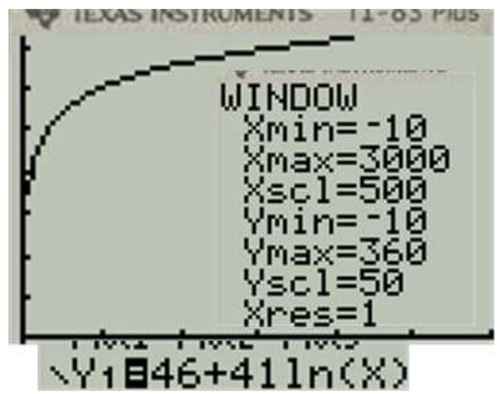
98. Alternately the problem could readily be solved by graphing!

Technically speaking Applied Math students need only know how to do graphical solutions and with the aid of graphing tools.



99. **Example - Solve a Ln Equation**. The altitude of an airplane is measured in 100's of feet in what are called **Flight Levels** (FL). Thus if an aircraft is at **FL350** it is at **350 *hundreds*** of feet, so **35,000** feet in altitude. Aircraft tend to climb to the highest optimum cruise altitude and of course as they burn off fuel weight they can *slowly* climb higher. The cruise altitude, **A**, in units of **FL** of a particular aircraft is a function of the distance, **d**, in miles it flys and is modelled by **A = 46 + 41ln(d)**.

a. Graph the logarithmic curve of the height vs distance to get this→



b. Evaluate the cruise level formula to calculate at what altitude the aircraft should be after 2500 miles. **A =** ____________

b. using a graphing tool calculate at what altitude, A, the aircraft should be after 2500 miles. **A =** _______________

c. solve to find the distance flown if the aircraft is at FL330. **d =** ___________

APPLICATION OF LOGARITHMS

100. We use LOGS when we talk about earthquakes (The Richter Scale), when we do Chemistry (the $\mathbf{P_h}$ or strength of an acid), when we measure volumes of sounds (Decibels), or brightness of stars or calculate the age of an ancient artefact using carbon dating. In all these cases we don't talk about the amount of destruction or volume or acidic concentration strength, we talk about the *exponent of the amount* of those, otherwise the actual number would be way to big or way to small to write and would become very cumbersome.

101. **Sound Volume – Decibels**. Invented by Alexander Graham Bell, an honorary Mohawk! Since sound can be so very quiet or so very loud he found it convenient just to talk about the logarithm (the exponent) of the sound level. A cat purring at one metre away has a sound level of about 25 decibels. This would be the exponent of the sound power reaching your eardrum in watts per square cm on your ear drum (or something like that which is not important here). To slightly complicate things, Alexander Graham Bell did not like numbers that had decimals in them, so when he said 25 decibels (dB), he meant 25 tenths (a *deci*-) of a Bel; so 2.5 Bels.

102. Regardless of the technicalities, a sound that is 20 decibels louder than another sound is actually doing 100 times as much damage to your ear! Since 20 tenths of a Bel , is 2 Bels which is an exponent of 2 on a base 10 Log and $\mathbf{10^2 = 100}$.

103. Most doctors will say that prolonged exposure to an 85 dB sound will quickly cause ear damage. How much more damage are you doing if you have your music cranked up to 100dB?

104. $Power\ increase = 15db$; = 1.5 bels.   But a logarithm is an exponent. Power increase = $\mathbf{10^{1.5}}$ = **31.6 *times as loud***!

105. The same ideas apply to earthquakes and the Richter scale of an earthquake strength . An earthquake of magnitude **7.6** is how much more deadly than an earthquake of magnitude **6.3**? $\mathbf{10^{(7.6-6.3)} = 10^{1.3} = 20}$ times stronger!

105. Star brightness (magnitude) is measured in logarithms too! Except **not** with a **base of 10**! Star brightness is measure in logs whose base is **2.5**! Not only that but the intensity is arranged so that big brightness numbers are dimmer. So a star with an intensity of 4.2 is brighter than a

star of magnitude 6.8 by a brightness amount of 2.6 magnitudes which is really in an absolute brightness sense $2.5^{2.6}$ times brighter or about 11 times brighter to your eye.

106. So you can explore those types of measurements on your own, but you see that we use logarithms almost everyday of our lives.

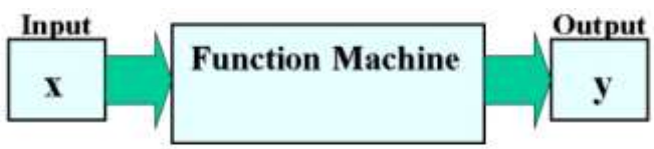
APPENDIX A
TO GRADE 12 UNIT C NOTES
FUNCTIONS

HASTY REVIEW (INTRODUCTION) OF FUNCTIONS

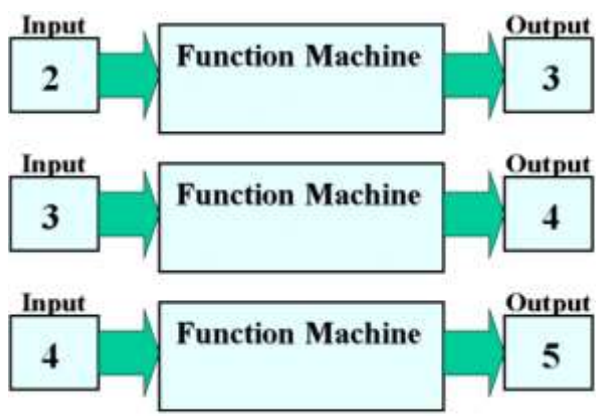
This material would have been covered in Grade 10 and again in Grade 11 Applied!

FUNCTION MODELS

1. A function model is the final way to think about the more common relationships between numbers. Using function models we think about a machine that takes an **input** number, **x**, and spits out an **output** number, **y**. It really is as simple as the diagram below:



2. Can you guess what this machine does to an input **x**? Let's call it the **F** machine for short.



Answer:

(Ans: it adds one to the input.
$f(x) = x + 1$.

3. ***f(x)* Notation**. The 'F' machine above took the input of 4 and did a function on it to output the number 5. The process could be written as f(4) = 5 because all the machine does is add 1 to the input number. It is pronounced 'f at 4', or ' f of 4 ' is 5. In the case above f(5) = 6. So what is **f(8)**? Answer: ____________. How did you calculate **f(8)** ? _________________.

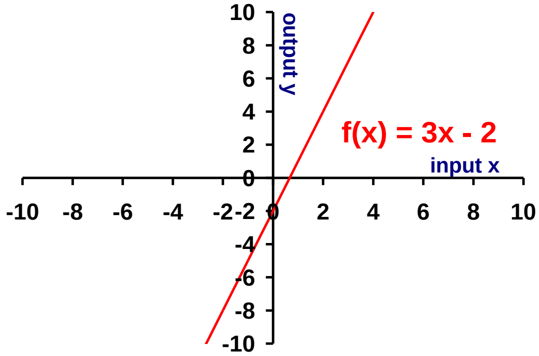
Be very certain: **f(x)** ***does not mean*** some unknown **f** multiplied by some unknown **x.** It just means there is a function that does something to an input of **x**. In fact, it can be **G** machine if you want; ***g(x)*** or whatever you name you want to call it.

4. Here are several more functions to **evaluate**:

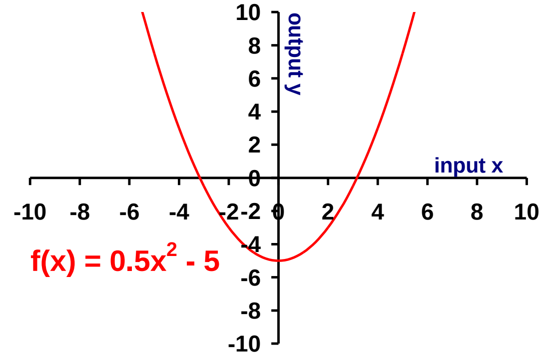
Input: x	**Function Process**	**Output:f(x)**
2	**f(x) = 3x + 1**	
½	**f(x) = 3x + 1**	
–3	**f(x) = 3x + 1**	
2.5	**f(x) = x^2 – 4**	
$\sqrt{4}$	**f(x) = x^2 – 4**	
0	**f(x) = x^2 + 2x – 5**	
2	**f(x) = x^2 + 2x – 5**	
7	**f(x) = 4**	
–2	**f(x) = 4**	

Recall from Grade 9; 'evaluate' means to plug in values for unknowns to find a value for the entire mathematical expression

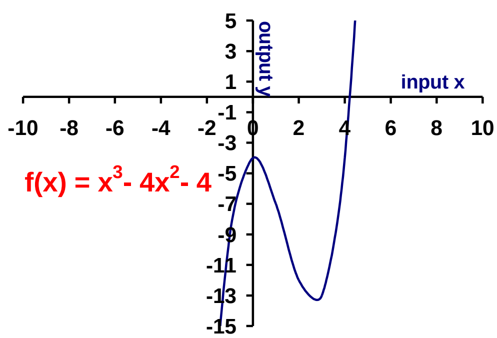
5. Normally we plot the ***input*** of a function on the ***x-axis*** and the ***output*** of a function on the ***y-axis*** of a graph. Below are some typical examples of the graphs of some functions. You can check them with your graphing tool if you want.



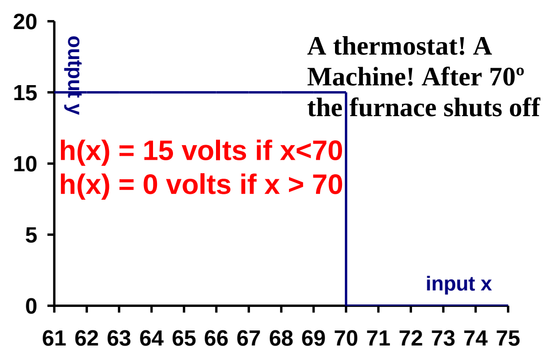
a. Function Graph A



b. Function Graph B



c. Function Graph C



d. Function Graph D. (does it matter if I call the process 'h' for heat instead of 'f'?)

6. **Domain and Range of Functions**. The Domain and Range of a function are often fairly simple and are determinable from the function expression and from the graph of the function.

7. **Domain of a Function**. Normally at this grade level the domain of function is all real values of ' **x** '. So, for example, in the four functions and graphs above we see that there is nothing to prevent us from inputting **any value** of ' **x** ' that we want into the machine, it will spit out an answer! So the domain for all of the above function graphs is just: **– { ∞ < x < ∞ }** .

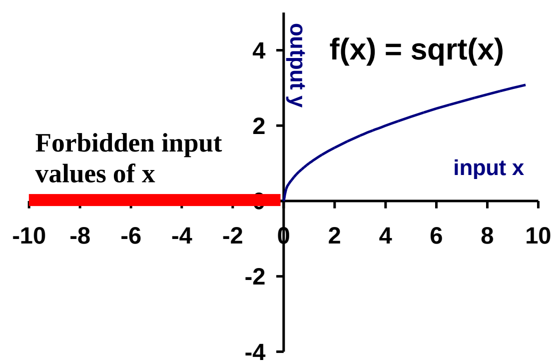
8. **Range of a Function**. Often the range of a function **f(x)**, its output being graphed on the y-axis, is limited in some manner though. The range of the four Function Graphs above are:

Function Graph A: $\{ -\infty < f(x) < +\infty \}$
Function Graph B: $\{ -5 \leq f(x) < +\infty \}$
Function Graph C: $\{ -\infty < f(x) < +\infty \}$
Function Graph D: $\{ 0 < f(x) \leq 15 \}$

9. **A Function with a Limited Domain**. For now, the only function you are familiar with that has a limited domain is the 'square root' function. The square root function would be written in function notation as: $f(x) = \sqrt{x}$.

10. Are you 'allowed' to do the square root of any number? No! You cannot do the square root of a negative number! So, the domain of $f(x) = \sqrt{x}$ is:

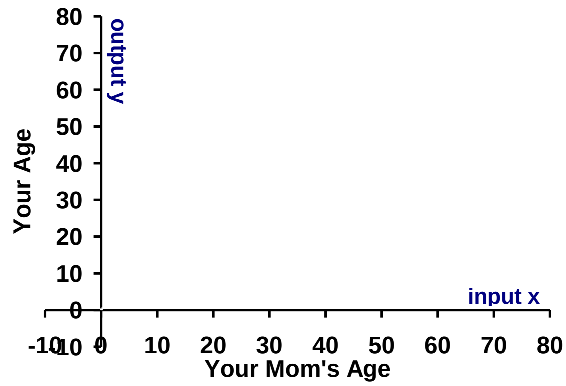
$\{ 0 \leq x < \infty \}$



11. As long as you input a number of zero ***or more***, this function machine will output a number. Anything less that zero and the 'machine' will blow up! (not really!)

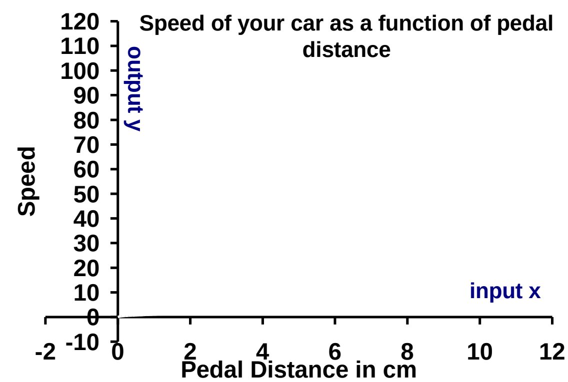
12. Sketch your age as a function of your mom's age.

What are the approximate Domain and Range of this functional relationship?



13. Sketch what the speed of your car as **a function of** how far you push down the gas pedal.

What is the domain and range of the function that models this?



Can you go backwards by pulling up on the gas pedal??

So that is a quick introduction and or review of the idea of functions. Check out the Grade 10 Applied resources if you want more.

APPENDIX B
TO GRADE 12 APPLIED FUNCTIONS
REGRESSION OF FUNCTIONS - DESMOS GRAPHING TOOL

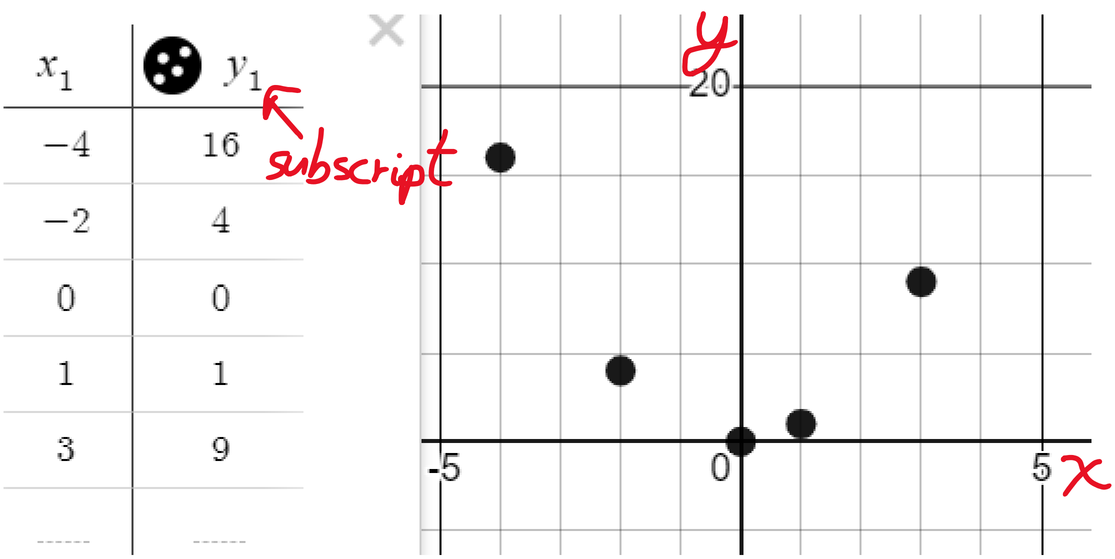
REGRESSION

1. **Regression** is a statistical method to find the equation (or a close equation) that expresses the relationship between two sets of values (like coffee temperature and time, or stopping distance and speed, or marks and attendance). Regressions can be done in EXCEL (using a *trend line* of data) or on a Graphing Calculator.

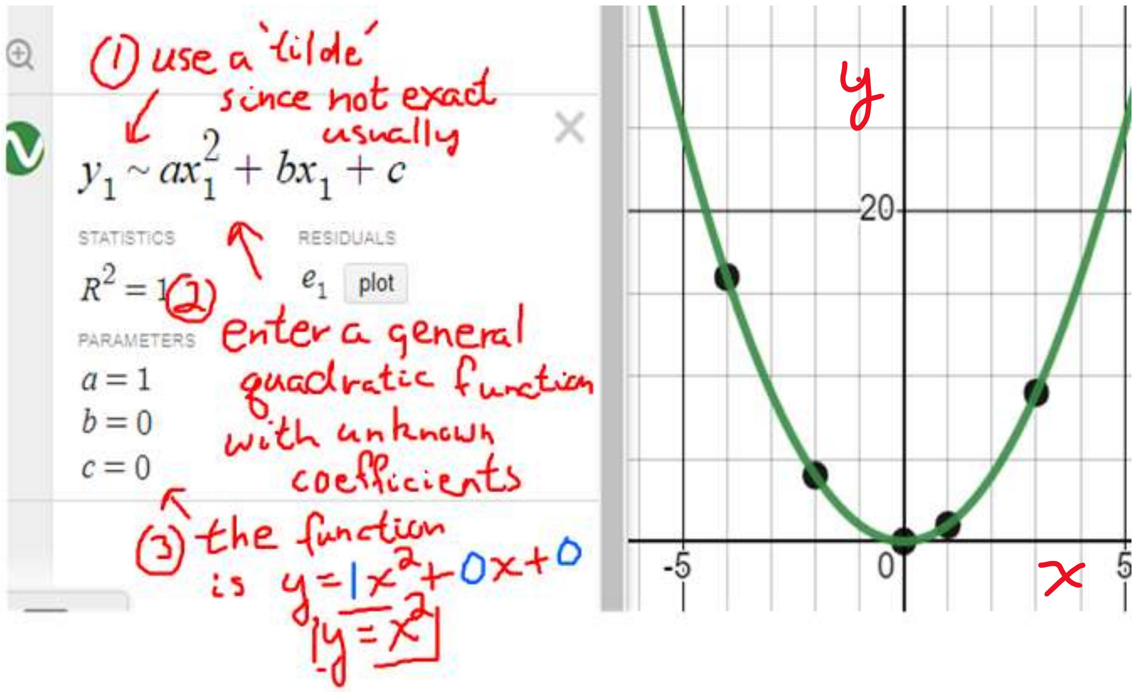
2. Let's do a regression on the following simple data: ie: let's find what equations best represents this data and how well.

x	-4	-2	0	1	3
y	16	4	0	1	9

3. Do scatter plot to get a rough idea of what function might be best.



Does that look familiar? Looks like one 'hump'.



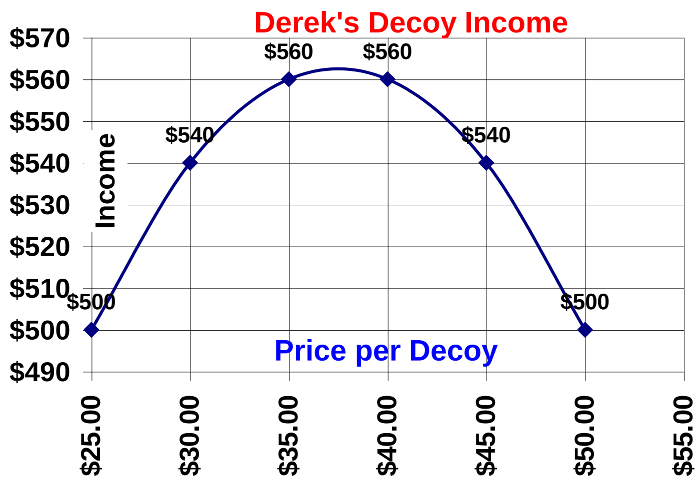
In this simple case (which you probably figured out without a regression) the function is $y = x^2$. The $R^2 = 1$ means it is a perfect fit!

Note that you could have tried fitting a cubic to this but it would tell you that the cubic term was 0. You could try fitting an exponential to this but the R^2 would indicate there would be no good fit.

5. **Example Regression**. Derek makes duck decoys every duck season! They are rather good! He sells them for **$25.00 each**! But he wonders if he should be charging more! When he charges **$25.00** he gets **20** customers every season. But he asked some questions and has determined that if he were to charge **$5.00 more** he will **lose two customers**. And he has determined that every **increase of $5.00** will lose him two further customers with each increase. He has been told by his Math teacher and a Business Commerce graduate that the relationship between selling price and income is quadratic. So what should Derek charge for each of his decoys to maximize his income?

Price	$25	$30	$35	$40	$45	$50
Customers	20	18	16			
Income	$500	$540				

6. **Plot** the scatter plot and use a quadratic regression on a suitable graphing tool to find the equation that relates Derek's **income** to the **price** he charges per duck. Your graph should look like this:



7. What is the function or formula that connects all the points (you need to enter at least three points for a quadratic)?

____

8. What is the price Derek should charge per duck to make maximum profit and what will that income be? (*ie*: where is the vertex)

____

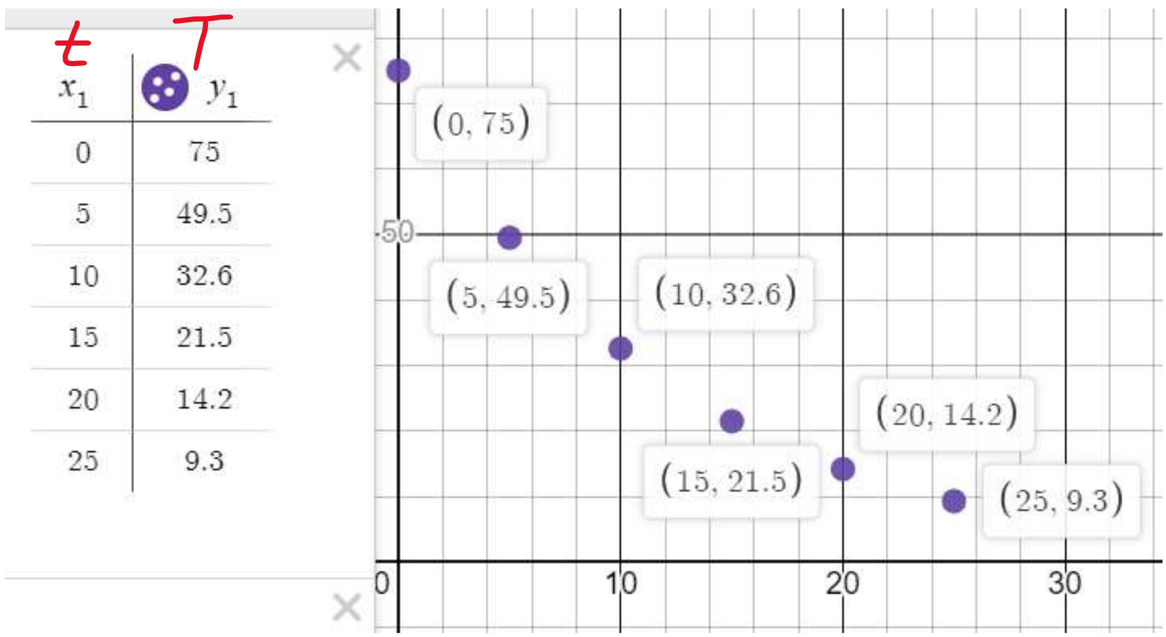
(You might be familiar with how to do this sort of stuff in EXCEL also)

REGRESSION OF AN EXPONENTIAL FUNCTION (Decay)
COFFEE COOLING

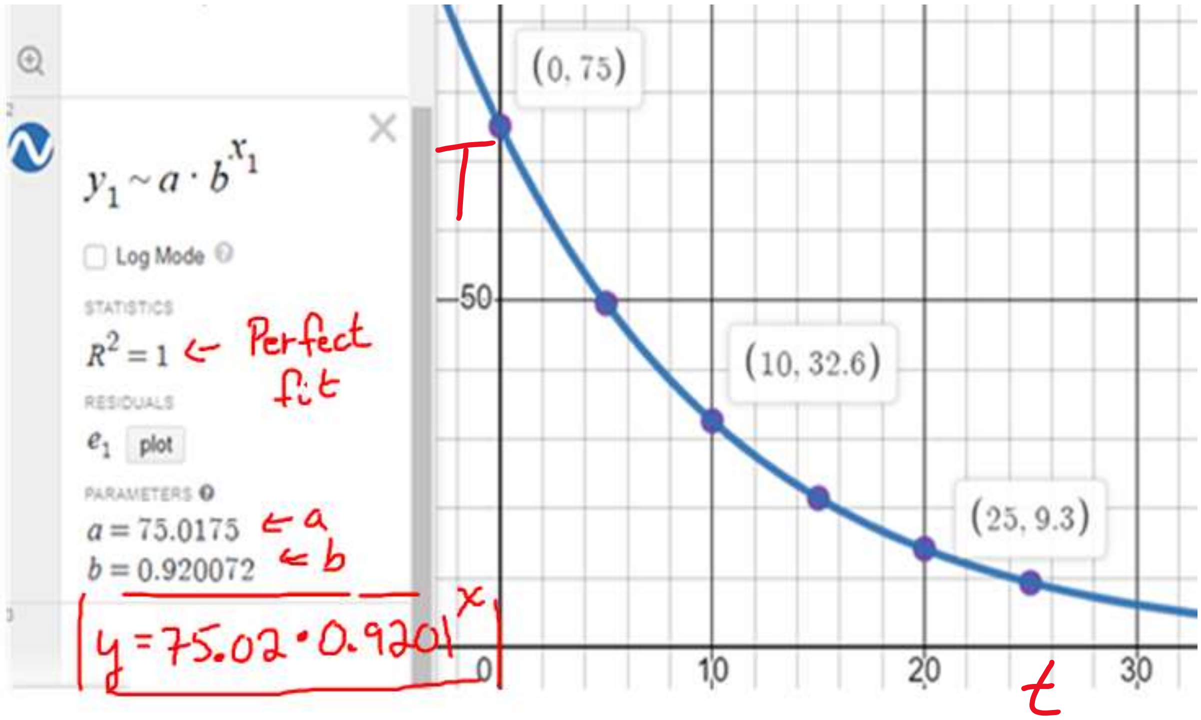
9. You are observing the temperature of your coffee as it cools; you collect the following data:

Time ,**t** [min]	0	5	10	15	20	25
Temp above room temp, **T** [°C]	75.0	49.5	32.6	21.5	14.2	9.3

Plot the scatter plot in the DESMOS graphing tool and determine the equation that governs the cooling of the coffee.



Now determine the regression equation that best fits this data. We suspect it is an exponential decay so try a general exponential function.



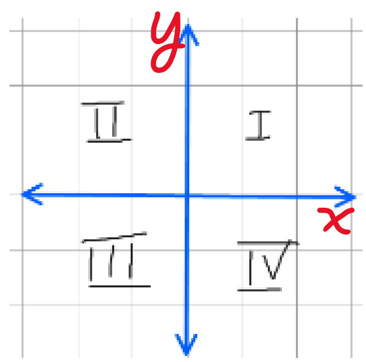
You could try a quadratic if you wanted and might get a reasonable fit as well for the domain in question.

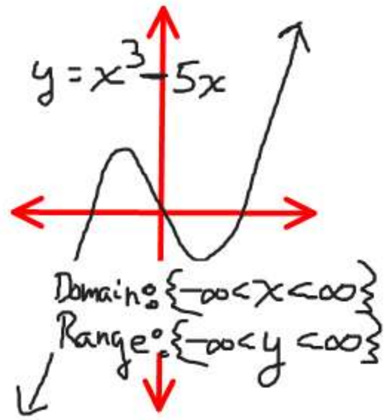
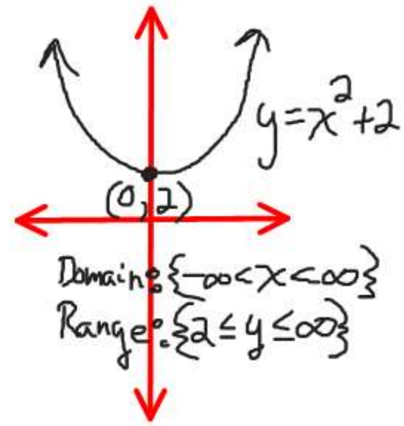
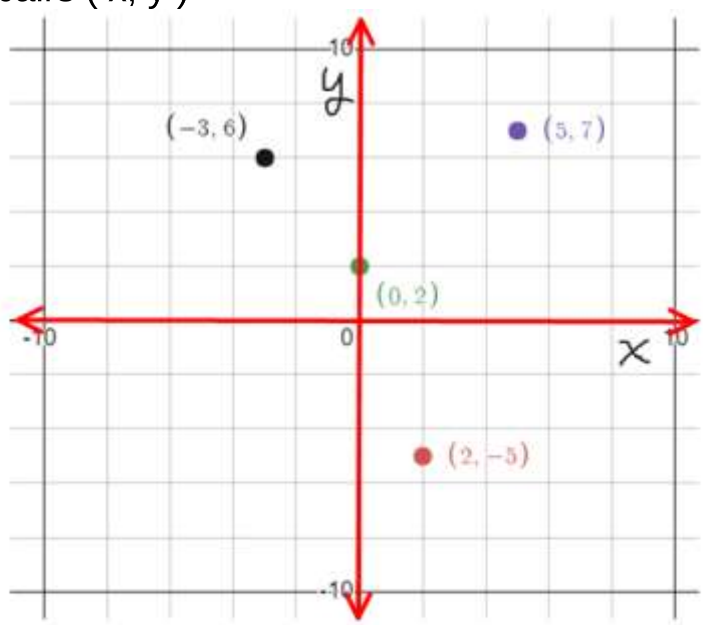
Coming up with a mathematical model to predict data is a very delicate operation as many will recall from the attempt at COVID virus predictions.

asymptote	A value that a function approaches but never quite reaches. 'my sister approaches cuteness asymptotically' *lol* Asymptotes are characteristic of exponential functions. 
coefficient	a value that amplifies the effect of some variable 3*x has a coefficient of 3 The '*x*' is 'amplified' by a factor of 3 The expression **ax^3+bx^2+cx+d** has coefficients of a, b, and c. The 'd' term has no variable with it, the 'd' is just a **constant** value.
Domain (of a function)	The allowed values of *x* in a function. The allowed inputs. For most of the functions we study the *x* can be any value between negative infinity and positive infinity $\{-\infty < x < +\infty\}$ Given that one cannot actually get to infinity it is necessary to use the inequality symbols.

exponent	An exponent indicated how many of a base number, b, are multiplied together. Example: 3^4 means four 3's multiplied together, ie: 3*3*3*3 = 81 Sometimes indicated a 3 ^ 4
Exponent laws	Recall Grade 9 exponent laws $b^m * b^n = b^{(m+n)}$ $b^m/b^n = b^{(m-n)}$ $(b^m)^n = b^{(m*n)}$ $b^{\frac{1}{m}} = \sqrt[m]{b}$ $b^{-x} = \frac{1}{b^{+x}}$ $5^2 \cdot 5^4 = 5^6$ $5^6/5^4 = 5^2$ $(3^2)^4 = 3^8$ $b^{1/2} = \sqrt[2]{b}$; $9^{(1/2)} = \sqrt{9} = 3$ $2^{-3} = 1/2^3 = 1/8$
function	An expression that states how one variable changes with respect to another. My age = moms age plus 20 y = x + 20 The height, h, of an arrow [in metres] as a function of time, t, [in seconds], is **$h = -5 * t^2 + 50t + 2$** The earth's population is exploding exponentially as: **$P = 8 * e^{0.012 * t}$**, P is the population in Billions and t is years.

Integer operations	Knowing how to handle signed values (negative and positive) is important! Adding opposite and negative values. The sense having the larger amount 'wins'! 4 + (–9) = –5 –3 + 5 = +2 Turn all subtraction into adding the opposite sense: 4 – 2 = 4 + (–2) = 2. 4 – (–2) = 4 + (+2) = 6 Multiply and Divide signed values Multiply or Divide **like** signs is positive Multiply and divide **unlike** signs is negative. 5*(⁻3) = ⁻15; ⁻6* (⁻8) = +48; ⁻15 / ⁻3 = +5; $^{+48}/_{-8} = -6$
linear	Data that makes a line on a scatter plot.
polynomial	A collection of terms of powers of a variable. Eg: $ax^3 – bx^1 + 2$ is a general third degree polynomial.

Quadratic function	A function involving a square of some values as the highest degree. There are several forms The general form: y = x^2 – 5x + 6 The vertex form: $y = \left(x - \frac{5}{2}\right)^2 + \frac{7}{2}$ The factored form: y = (x – 3) * (x – 2) All three above are the exact same relationship just expressed differently. We will tend to favour the general form.
quadrant	The four quadrants of a cartesian grid. Usually indicated by Roman Numerals. 

range (of a function)	The subsequent permissible values of a function. Some functions have a peak (maximum or minimum) limit. Some functions are unlimited and the output values are between negative infinity and positive infinity Some functions level off at an asymptote.  
scatter plot	A plot on the cartesian grid of points given by ordered pairs (x, y) 

sketch	To make a rough representation of a relation that preserves all of the significant points such as maximums, minimums, and intercepts.