GRADE 12 ESSENTIAL
UNIT I – PROBABILITY

CLASS NOTES

Introduction

1. You use probability all the time. Should you cross the street against the red hand? If there is a chance of rain should you wear your suede jacket? Will your bus be a few minutes late so you still have time to catch it?

2. If you don't know what will happen on any one event but you do know what will happen over many events, that is using probability.

3. For example; if I get you to go to the parking lot and see if the first car you see has two numbers the same on its licence plate I can't predict whether it will or it won't. But I can tell you with really high confidence that if you sampled 100 licence plates then 27 of them (give or take one) would have exactly two digits the same. (28 if you were looking for two *or more digits* the same on the licence plate)

Probability is just a way of showing what portion of your attempts are *successful* with the desired *outcome*.

EXPERIMENTAL PROBABILITY

4. You have already been studying some elements of probability with your study of Statistics when you took sample measurements and looked at how they were distributed, what percentage of Prairie Dogs were more than 40 cm and what were less than less than 40 cm for example.

5. Take a plastic spoon and flip it 50 times on onto a table surface. Record whether it falls cup 'up' our cup 'down'.

What percentage of the flips did the spoon land up?

What percentage did it land down?

Outcome	Tally	Count	Prob
Up			
Down			
	Totals:		

6. Calculate:

Probability= $\frac{Number\ of\ Favoured\ Outcome}{Total\ Number\ of\ Trials}$

Prob(Up) = $\frac{\boldsymbol{Number\ 'Ups'}}{\boldsymbol{50}}$ =

Prob(Down) = $\frac{\boldsymbol{Number\ 'Downs'}}{\boldsymbol{50}}$

above, we conducted an experiment and calculated an experimental probability for a particular outcome. Each flip was called a trial. Each trial in this case had only two possible outcomes in our experiment, either 'Up' or 'Down'.

7. If you conduct an experiment with a large enough number of trials you can be pretty sure that you will be able to get pretty close to the same probability percentage (under the same experimental conditions) if you conduct the experiment again and again.

(FYI: It is called the law of large numbers, the larger the number of samples you take or trials you use gets you closer to the real probability)

EXPRESSING PROBABILITY

8. Probability can be expressed as a fraction (a ratio), a decimal or a percent. To work in probability you have to be proficient in converting between all representations. If you are not familiar with how to convert between fractions and decimals and percentages manually then make sure you know how to use your calculator that does it. Or see your teacher.

Fraction (ratio)	Ratio Statement	Decimal	Percent
$^{1}/_{2}$			
$^{5}/_{8}$			
$^{13}/_{25}$			
	1 to 5		
	7 **:** 8		
		0.18	
		0.65	
		0.95	
			100%
			92%
			67%

THEORETICAL PROBABILITY

9. Often there is no need for an experiment. You can simply predict what will happen by counting what you are looking for.

10. **Example**: A bag contains 10 marbles. There are exactly **three** green marbles in the bag. What is the probability that if you randomly select a marble (eyes closed) it will be green?

Prob (Select Green) = $\frac{Number\ of\ Green\ Marbles}{Total\ Number\ of\ Marbles} = \frac{3}{10} = 0.3 = 30\%$

This calculation requires completely '**random**' outcomes of course. By random we mean that each of the 10 marbles is equally likely to be drawn; someone hasn't glued a marble to the bottom of the bag or some marbles are not smaller than others, etc. In other words there are **no favoured outcomes**, the drawing is '**fair**'.

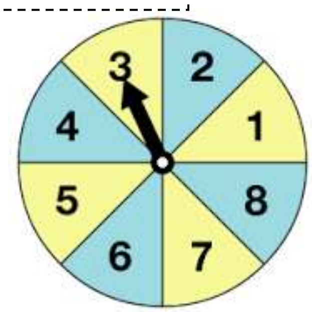
11. If someone conducts an experiment (or a survey) and it doesn't get the same probability as the theory, then that is a good clue that someone has been messing around with the possible outcomes (or messing with the survey data)! Assuming of course the experiment involved a large sufficient number of trials or samples.

SAMPLE SPACE

12. A sample space, **S**, is a list of all the possible outcomes. So, the list of all the possible outcomes for rolling a dice is: **{1, 2, 3, 4, 5, 6}**. We call the sample space the set '**S**' since we are to lazy to write the phrase 'sample space'. The size of the sample space, **n(S)**, '*number in the sample space S*', is 6. There are six possible outcomes and they are all equally likely.

13. **Example**: Karen and John are thinking of starting a family. They want two children. What is the sample space for the gender arrangements of the two children that they could have. List the possible outcomes of their children.

14. **Example**.



a. What is the sample space, **S**, of outcomes on this spinner? List them: ____________________

b. What is the size of the sample Space, **n(S)** ?

______

c. Calculate the probability of spinning a seven on one spin!

Prob (Spin 7) = $\frac{\mathbf{n(sevens)}}{\mathbf{n(S)}}$=

d. what is the probability that if you spin again that you will get another seven?

FYI: The first and the second spins are called independent events, the first event did not affect the second event. But we do not study that in Essential Math. Just in case you see it in a reference somewhere

15. Calculate the probability of spinning an even number:

Prob (Even) =

Notice that these type of simple calculations require that the each outcome in the sample space be equally likely. If this was an 'unfair' spinner that could be made to favour a certain outcome (like at some fairs and exhibitions!) the theoretical probability calculations would not be reliable.

PROBABILITY OF SOMETHING NOT HAPPENING

16. Have you ever noticed that if there is a 10% probability of snow that means there is a 90% probability of no snow?

Have you ever noticed that if the probability of drawing a King from a deck of cards is 4/52, then the probability of **NOT** drawing a King is 48/52? Or more correctly 12/13, since we always reduce fractions.

17. Those types of outcomes, where it is or it isn't, are called '***complementary***'; they add up to 100%. An outcome happens or it does not happen! If we call the outcome 'A' and we want the probability of 'NOT A', we can say:

Prob (NOT A) = 100% – Prob (A) if working in percents
or
$Prob(\overline{A}) = 1 - Prob(A)$ if working decimals or fractions
or
$$Prob(\overline{A}) = \frac{\boldsymbol{n(\overline{A})}}{\boldsymbol{n(trials)}} = \frac{\boldsymbol{n(trials) - n(A)}}{\boldsymbol{n(trials)}}$$

Notice in the last statements that mathematicians are even too lazy to write 'Not A' they just put a bar over top of the outcome to say it is the complement of the outcome. $'\overline{A}'\ is\ the\ complement\ of\ outcome\ A$

18. Examples:

a. what is the complement of the event: 'rain'? ___________

b. what is the complement of the event: drawing a green marble from a bag? _________

19. Notice I have snuck in the word 'event'. Sometimes a desired outcome is a combination of outcomes so we call it an event. So drawing a Black Queen from a deck of cards would be an event because you want a card that is Black and it is also a Queen. So you will see the term '**event**' instead of outcome fairly often and used interchangeably.

20. Another way you can say this is that the probability of something happening added to the probability of it not happening is 100%

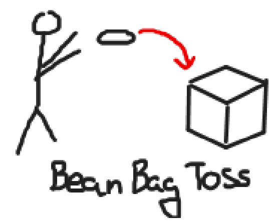
$$\boldsymbol{P(A) + P(\overline{A}) = 1}$$

Expected Number of Outcomes

21. We have the fancy formula $\boldsymbol{P(A) = \frac{n(A)}{n(Total)}}$; the number of favoured events A divided by the number of all equally likely possible events.

We can use that to predict what we expect when we do multiple trials.

Example: if the probability of getting a bean bag 'ringer' in a box from 4 metres away is 20%, then how many ringers *should* we ***expect*** get if we throw the bean bag 50 times?



$\frac{\mathbf{20\ ringers}}{\mathbf{100\ throws}} = \frac{\boldsymbol{x}\ \mathbf{ringers}}{\mathbf{50\ throws}}$; ∴ **x** = 10 ringers

or we could just re-arrange the probability formula for probability as follows: For favoured event 'A'.

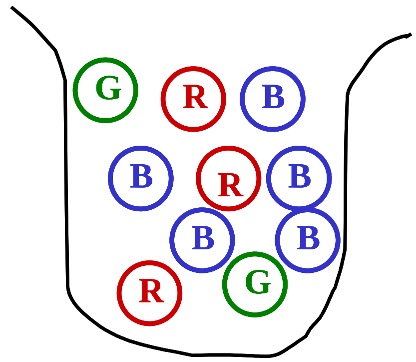
n(A) = P(A) * n(total trials); so n(A) = $\frac{20}{100} * 50 = 10$

22. Example: If there is a recorded 15% chance of rain on any day in July, then how many days *should* we *expect* it rain in July? (we have been keeping track of these statistics for over 100 years!)

Notice you can have fractions of a day as an answer since this is an average result for many Julys.

23. **Example**: A bag of marbles contains five blue marbles, three red marbles and two green marbles. You reach into the bag and draw out one marble. What is the probability it is green?

P(Draw Green) =



25. That was pretty simple probability for one draw of a marble. But what is the probability if you draw out another marble that it will be green again?

That type of calculation is called a Compound Probability problem and we do not do it in Essential Math, I showed you just in case you see it in a reference somewhere related to what is called Dependent Probability, an event which is affected by a previous event.

26. If you throw the letters of TABASCO into a hat and make a single draw, what on that one draw is the:

a. Prob(Letter O):

b. Prob (Letter A):

c. Prob (Letter P):

ODDS FOR AND ODDS AGAINST

27. Occasionally you will encounter the concept of 'Odds'. Someone will say: 'What are the odds of that happening?'

Odds are another way to express the likelihood of something happening except instead of comparing what you favour to everything that is possible ; you compare what you favour compared to what you don't favour (or vice versa). You compare wins to losses, successes to failures.

28. There are **two types** of '**Odds**'. There is 'Odds For' and 'Odds Against'.

ODDS IN FAVOUR

29. Odds **FOR**, or in favour of, an event A are calculated as
Odds For Event A are shown as $n(A) : n(\overline{A})$; the number of ways for event A **to happen** as a ratio with the number of ways for A **to not happen**.

So, for example if the probability of a successful bean bag toss is 2/9; then the Odds in Favour (or for) a successful bean bag toss are 2:7 or we simply say the Odds in Favour, or the Odds For, a successful toss are 'two to seven'.

ODDS AGAINST

30. Odds Against an event happening are simply the reciprocal statement.
Odds against an event, A, are stated as: $\overline{\boldsymbol{A}} : \boldsymbol{A}$

We don't normally write any Odds as a fraction; that would be confusing. We usually use the colon notation; ':' or just say it in words.

Example: If the probability of the Leafs winning tonight's game is 25%; then the **Odds Against Winning** are 3 to 1.

31. **You Try**. The probability of a snowstorm tonight is 35%. Determine the **Odds Against** a snow storm tonight.

Ans: 13 to 7 Against a snowstorm

30. **Gambling Example**: At the horse race track the likelihood of a horse winning is tracked statistically. At the race track they give the likelihood of a particular horse winning as Odds Against Winning!

So, a horse that has '**Betting Odds**' of 10:1 has a 1 in 11 **probability** of winning; or approximately a 9% probability of winning. If you are familiar with how Sports Select works (hopefully not) then you are familiar with how your payout is based on the Odds Against Winning. If you were to bet $2 on a horse with 10:1 odds and it actually did win, despite it was unlikely, you would get back $20. We will study gambling in our **expected value** section.

33. Complete the blanks in the table:

Probability	**Odds in Favour**	**Odds Against**
50%		
30%		
	1 to 5	
2/5		
	2 to 3	
		5 : 2
		8 to 3

EXPECTED VALUE:

34. ‘**Expected value**’ is a key element of gambling but in reality we gamble with more than money. The basic idea is to answer whether it is worth risking something to make an overall gain.

Is it worth crossing against the red hand at the cross walk? What loss are we risking, what is the likelihood we gain? Risk vs reward, we do it all the time!

35. A famous Mathematician , Blaise Pascal [1623 – 1662] explored probability and gambling. He reasoned that a rational person would be believe in God because what they risk (being wrong) is trivial compared to what they gain (eternal happiness in heaven) regardless of how unlikely it is that there actually is a God. So he considered it a good wager.

Of course you cannot put a number value on eternal bliss, so the math is a bit fuzzy.

Using the Expected Value (Is it worth the Risk?)

36. If you are asked to bet $1.00 for 10% chance to win $20.00, is that a good risk?

It turns out it is a wonderful thing! Or not? If you play 10 times you would contribute $9 to losing tickets, but 10% of those 10 times, ie: once, you should *expect, normally,* to win one time for $20 (less the one it costs you). Spending $10 to make $20 sounds awfully good. The Expected Value is +1.00 for each play. For every $1 you risk you get it plus another dollar back (on average after many plays).

38. Let’s examine another example more closely.

Say you have a 1/5 chance of winning $4 by buying a $1.00 ticket.

If you play 5 times you would *expect* to win once so you would have a WIN of $4 less the $1.00 it costs to play for a net Gain of $3.00. However, the other 4 times (the 4 out of times you lost) you had a net loss of $4.00. So, you have a net loss of $1.00 over the five times you played, so you *lose* on average $0.20 per play. Playing a game where you can expect to lose is a bit silly.

40. Look at it in this table:

Event	Probability	Amount Won	Cost to play	Net Gain or Loss [Payout]	Probability x Payoff
Win	1/5	4	1	$4 – $1 = $3	1/5 *$3 = $3/5 = $0.60
Lose	4/5	0	1	$1	4/5 * $1= $4/5 = $0.80
				EV	$0.60–$0.80= -$0.20 per play

THE EXPECTED VALUE FORMULA

41. The Expected Value Formula is given as:

EV = Prob(Win)*$ Net Gain – Prob(Lose)*$ Lost;
where '*net gain*' means the prize amount less the cost to win it.

a. A **positive value** for the EV means it is a good bet.

b. A **zero value** means it is neither a win nor loss so that after many games you break even.

c. A **negative value** means you are certain to lose money *on average* every bet. (unless of course 'you' are the casino!)

There exist no real situations where any gambling game would give a positive expected value unless you are the '*house*'. The only way to win at gambling is to own the casino. Everyone else's negative expected value return is your positive if you own the casino.

Advanced Example:

43. Say you have a dice game, a single six-sided die. It doesn't cost anything to roll the die. A '1' wins you $2, a '6' wins you $1, and any other roll wins nothing and you must pay $1.00. What is the Expected Value of this game?

Event	Probability	Amount Won	Cost to play	Net Gain or Loss [Payoff]	Probability x Payoff
Win $2	1/6	$2	0	$2	1/6 *$2 = $1/3
Win $1	1/6	1	0	$1	1/6* $1= $1/6
Lose	4/6	0	1	1	4/6 * 1= $^{-}$4/6
				EV=	1/2 – 2/3 = $^{-}$1/6 = $^{-}$$0.17

44. So you can expect to lose 17 cents every roll on average.

45. The insurance industry is based on gambling!

If you have life insurance you are gambling against whether you live or die prematurely. If you are a smoker there is a higher risk you will die pre-maturely , so you pay more premium to 'play' the game.

If you have car insurance you are gambling against whether you will have an accident or not. If you are a bad driver you will pay more to play the game since the probably of having an accident is higher. Further, if you have an expensive car you will pay more because the cost of a loss is more. If you drive a motorcycle you likely already know that insurance is very expensive. And in fact it can also affect your life insurance.

Of course if you drive 'under the influence' and have an accident your payoff will likely be zero as well.

If you have house insurance, you gamble against something happening. Depending on which part of town a homeowner lives the chance of damage or destruction is higher and the insurance premium will be higher.

So, you see that Expected Value is something you encounter all the time if you are familiar with insurance.

PROBABILITY GLOSSARY

A	
Arrangement	The manner in which things are organized. Sometimes the order is important, sometimes it is not.

C	
Combination *Not studied in Essential Math*	The number of ways of selecting objects is a combination. (order is not important) Example: how many ways can 3 players on a team be selected for a league all-star game? It doesn't matter what order you get selected in, just that you make the team. The answer is $_{10}C_3$ or 120 ways Compare to permutation.
Complementary Events	If an event will definitely occur, then the probability of its occurrence is one. As a consequence, the probability of an event A *not* occurring, written **P(~A)**, is the difference between 1 and P(A) or **1 - P(A).** The complement of an event **A** is also shown sometimes as $\bar{A}$ **.** $\bar{A}$ and **~A** are read as**: 'Not A'**

D	
Dependent *Not studied in Essential Math*	Event A and event B are **dependent** events if the outcomes of event A influence the outcomes of event B
E	
Event	A set of outcomes, one or more outcomes of an experiment. A specific outcome or type of outcome. A subset of the sample space. **Example**: the outcome of drawing a single card is 52 possible outcomes, the event of drawing a **red** card is a set of 26 of those. The sample space is all equally likely 52 cards of the deck, the subset is the event (RED CARD)

Expected value	An estimate of the average expected return or loss you will have. Expected Value (**EV**) = P(win) * (Gain) – P(lose)* (Loss) **Example**: If you have a probability of 20% of Winning $10.00 and for an investment of $4.00. EV = 0.2*(10 – 4) – 0.8 * (4) = 1.2 – 3.2= $^{-}$2.0 If the EV is <0 you can expect to lose that amount every play; If the EV = 0 you will break even; and If the EV > 0 you can expect to gain that amount every play. So in the above game you can expect to lose. If you play 100 times you can expect to lose 2.0*100 = $200.00
Experimental Probability	The chance of an event happening based upon repeated testing or observed trials. It is necessary to calculate experimental (empirical) probability when you cannot be sure of equally likely outcomes.

F	
Factorial Notation	In general, n factorial is n! = (n)(n – 1)(n – 2) ... (3)(2)(1). Note that 0! ≡ 1
Fundamental Counting Principal	Suppose that an event **K** can occur in ***k*** number of ways and after it has occurred, event **M** can occur in ***m*** ways. Thus, the number of ways in which both **K** and **M** can occur is ***k*m*** ways.

I	
Independent Events *Not studied in Essential Math*	Events A and B are independent if the probability of A is not influenced by the probability of B. Two events A and B are independent if P(A) = P(A | B) or P(B) = P(B | A).

M

Mutually Exclusive Events
Not studied in Essential Math

When two events cannot occur at the same time because they do not have any common outcomes these events are said to be mutually exclusive.

Example: The events of drawing a red card and the event of drawing a spade, are mutually exclusive. They are mutually exclusive because there are no red spades ♠.

O

Outcome

A possible result of a single trial of an experiment, a possible answer to a survey question.

EG: for the experiment of tossing a six-sided dice the possible outcomes are 1,2,3,4,5,6.

The entire collection of outcomes possible forms the sample space

Odds Against

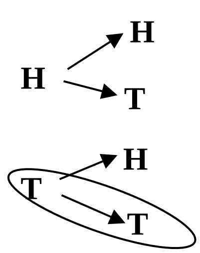
Ratio of unfavourable outcomes to favourable outcomes.
Example: The **probability** of rolling a sum of 12 with 2 dice is 1/36. Therefore the odds ***against*** rolling a 12 are **35:1**. Because there are 35 ways to **not** roll a 12 and only 1 way to roll it. Normally odds are usually expressed as *odds in favour* though.

Odds in Favour

ratio of favourable outcomes to unfavourable outcomes
Note that a probability of drawing a Heart from a deck of cards is 13/52 = ¼. The odds in favour are therefore **13: 39** or **1:3**

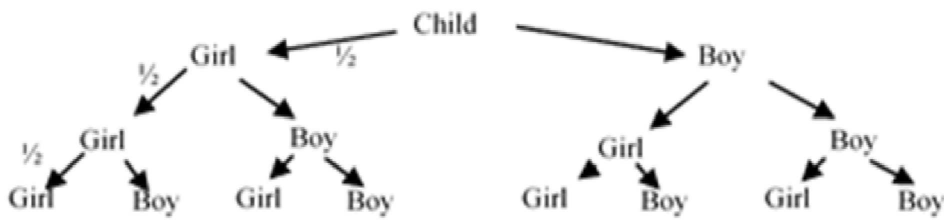
Outcome Tree

A tree showing all possible outcomes. This can get to be very big.
Example: All possible outcomes of flipping two coins:



There are four possible outcomes. Only one is Tails, Tails. Therefore probability of rolling TT is 1 in 4 or 0.25 or 25%

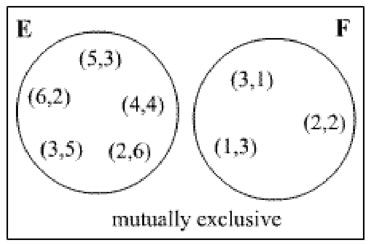
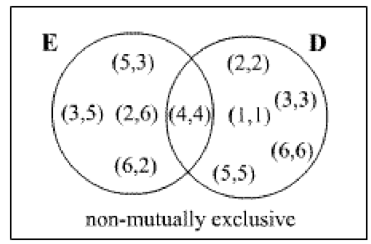
P	
Permutation	The number of ways of **selecting** and **ordering** objects is a permutation (two actions). Example: how many ways from 10 players from a team be ***selected in order*** for a first, second, and third prize? Answer is $_{10}P_3 = 720$
Probability	probability of the occurrence of Event A is the ratio of the number of occurrences of A to the total number of possible outcomes. Probability of an event is always 0 to 1 in decimal or 0 to 100% in percent.
Probability Tree	A tree showing all possible outcomes by probability. Usually better for doing probabilities. **Example** probability of having 3 daughters. The probability is ½ * ½ * ½= 1/8 = 12.5%



R	
Random	A random experiment is an experiment which can result in different outcomes, and for which the outcome is unknown in advance. There is no bias in the experiment.

S	
Sample Space	A sample space is the set of all possible outcomes of a trial or survey. **Examples**: ○ The sample space of rolling a normal six-sided dice is the outcomes: {**1, 2, 3, 4, 5, 6**}. ○ The sample space of tossing a single coin is the outcomes {**H, T**}. ○ The sample space of genders of three children in a family is {**GGG, GGB, GBG, GBB, BGG, BGB, BBG, BBB** }
Simulation	A mathematical experiment that approximates a real-world process. Example: You can simulate how many of each gender of child you will have by using a coin. Head is boy, tail is girl for example.

T	
Theoretical Probability	The chance of an event happening as determined by the mathematical result that it will occur.
Tree	Two types. An outcome tree to count all the possible items in a sample space or a probability tree when paths have different probabilities.
Trial	A sample; one of many tests to get a result in an experiment. A trial has an outcome.

V	
Venn Diagram *Not studied in Essential Math*	  A diagram that uses circles to show relationships among sets of numbers or objects.