GRADE 12 APPLIED
UNIT C – FUNCTIONS
CLASS NOTES FRAME

INTRODUCTION

1. In Grade 10 we studied lines. Each change in **x** had a proportional change in **y**. In **Grade 11** we will study more ‘***functions***’ (relationships between sets of numbers) that are not just simple ‘linear’ relationships. In Grade 11 you studied Quadratic Functions; functions with a ‘squared’ variable in the general form $\mathbf{y = ax^2 + bx + c}$.

What is a function?

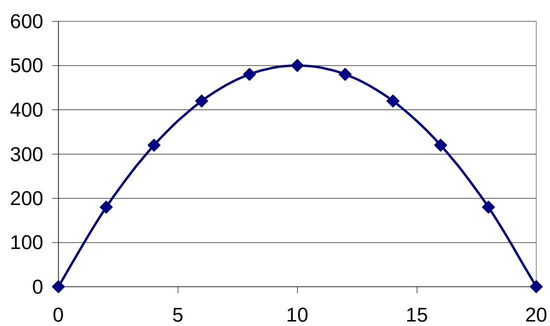
2. Go to the Appendix to these notes to review what a function is.

The next dozen pages are effectively a verbatim duplicate of Grade 11 Applied Quadratic Functions notes*.

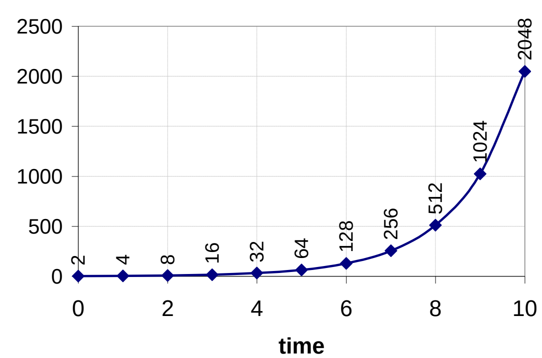
GRAPHS AND REAL LIFE

3. The best way to see a relationship between sets of numbers is to look at a graph. What might the following graphs represent? Or as we say: what might the graphs ‘model’ in the real world?

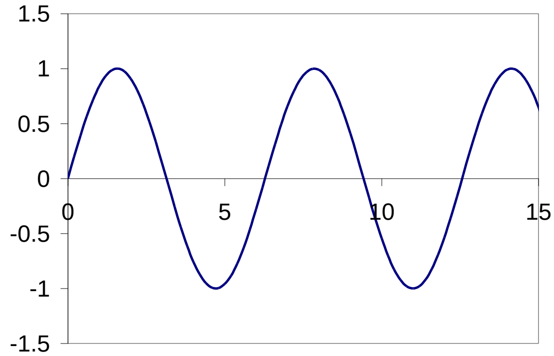
a.



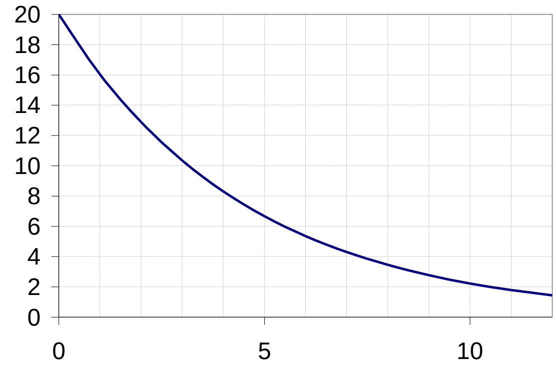
b.



c.



d.



DEGREES OF FUNCTIONS (we will better examine what a function is shortly)

4. We had already studied linear functions. They were a function (relationship) where the $y = mx + b$. So for example the **y** might be double the **x** and add **five**; $y = 2x + 5$. The x and y in the equation were just plain variables; no powers or square roots or reciprocals or anything fancy.

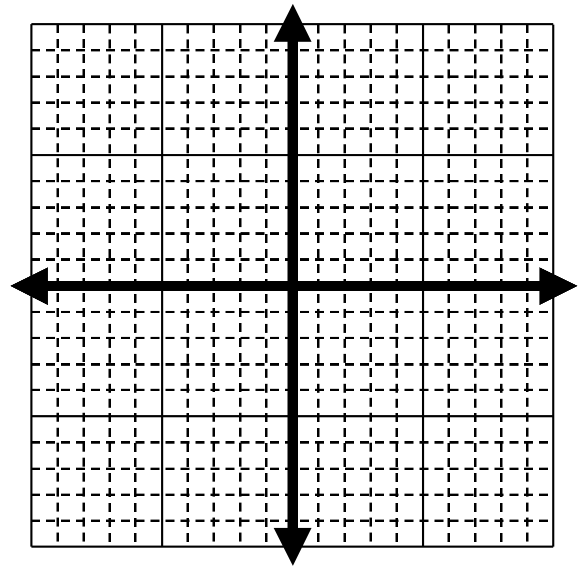
5. There are certainly other types of ***functions***. The next most important one is the ***quadratic function***. It is of '***degree 2***'. It looks like this: $y = ax^2 + bx + c$. Where the **a** and **b** (coefficients) and **c** (the constant), can be found to represent **real world** situations that relate the **x** numbers to the **y** numbers. Notice it is a polynomial expression, a bunch of powers of x added together.

6. The **degree** of a polynomial-type equation is the **highest exponent** of any variable in a function. So the equation of the function $y = ax^2 + bx + c$ is of **degree two**. It is called a ***quadratic*** function since '**quad**' means square in latin. Our former **linear** equation then was of **degree one**. $y = mx^1 + b$

CHARACTERISTICS OF THE QUADRATIC FUNCTION

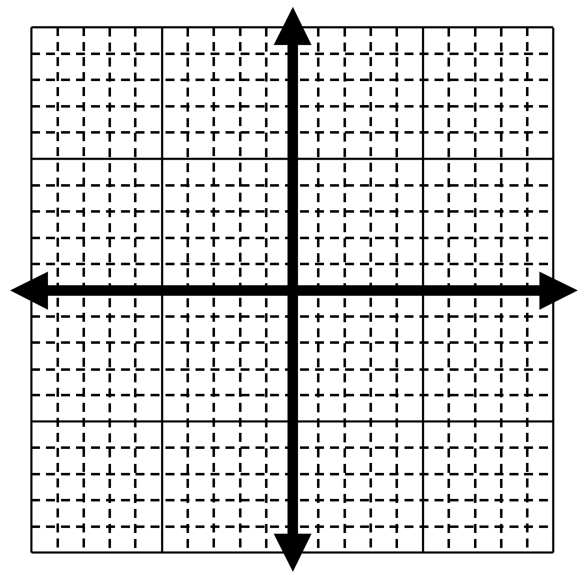
8. **Manually** graph the following two quadratic functions for given values of x in the associated table. You may find it necessary to make the scale of the *y*-axis different from the *x*-axis. (*Scaling* a graph)

a. $\boldsymbol{y = x^2 - 16}$



x	**y**
5	
3	
1	
0	
-1	
-3	
-5	
+6	
+8	

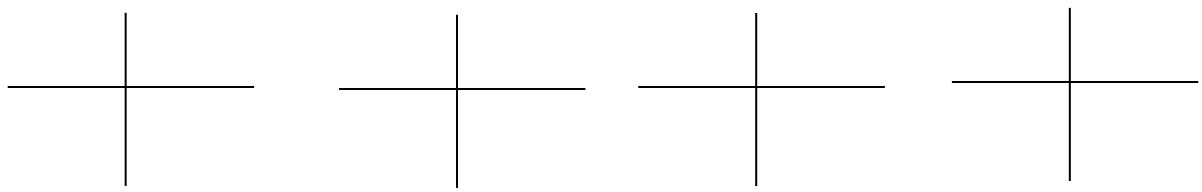
b. $\boldsymbol{y = -x^2 + 8x}$



x	**y**
-1	
0	
1	
3	
4	
5	
7	
9	
+8	

Now check both graphs with a graphing tool to see that it has been correctly done

9. Notice that a *quadratic* equation has one 'bump'. It also has a point where the function reaches a **maximum** (*or* a **minimum**). Also notice that a quadratic crosses the **y-axis** at exactly one place and one place only. It can cross the **x-axis** at **two places or less**. Also notice that there is **symmetry** to the quadratic curve called a 'parabola', the left half of the parabola is the same as the right side, just a mirror image. Invent a few quadratics of your own and graph them with a graphing tool and sketch the result below. See if you can find a quadratic function where it doesn't cross the x-axis at all! See if you find a quadratic function where it only touches the x-axis in one place! Write their equations and sketch them below: *If you are unfamiliar with the Texas Instruments TI-83 Graphing Calculator, turn to the Appendix at the back of these notes.*



10. *Remarks on sketching*. Notice when I ask for a *sketch* I do not give accurate graph paper with precise tick marks. All I want in a sketch is a rough representation of what a graph looks like. The curve should have its significant points in the correct *quadrant* however. Had I given you more exact graph paper I would expect significant points on the graph to be accurate.

11. **Vertex**. The **point** (***x, y***) where a quadratic reaches a **maximum** or a **minimum** is called the **vertex**. Use the TI-83 **CALC** feature to find the vertices now. What is the vertex for each of the two graphs above?

a. $\mathbf{y = x^2 - 16}$ (____, ____)

b. $\mathbf{y = -x^2 + 8x}$ (____, ____)

You will discover that when the a 'leading coefficient' is negative that the parabola opens down (cap) and when it is positive that the parabola opens up (cup)

12. **Line of Symmetry**. The line of symmetry is just the **vertical line, x = constant,** that passes through the vertex. What are the lines of symmetry for the curves above?

a. $\mathbf{y = x^2 - 16}$ **x =**__________

b. $\mathbf{y = -x^2 + 8x}$ **x =** __________

The vertex and the line of symmetry can also be found by: ' ***–b/2a'****; but that is Pre-Calculus information!*

13. **Domain and Range**. Recall from Grade 10 that **Domain** is all the values that the *input x* can have, and **Range** is all the subsequent values that the *output y* can have. Also notice that with a quadratic, because it has a ________________ or else a _________________ that the range will *never* be all numbers. There will be some numbers that a quadratic function can never attain. Find the domain and range of the two functions above.

a. **$y = x^2 - 16$**) Domain: ______< **x** <________
Range: ______< **f(x)** <________

b. **$y = -x^2 + 8x$** Domain: ______< **x** <________
Range: ______< **f(x)** <________

14. **Intercepts**. The intercepts are where a function crosses a coordinate grid axis. The *y*-intercept is where the function crosses the *y*-axis (that is, where $x = 0$) and the *x* intercept(*s*) are where the function touches the *x*-axis (ie: where $y = 0$). For a quadratic there can be either 0, 1, or 2 places where a function touches the *x* axis. Use the TI-83 **CALC** feature to find the intercepts now.

a. **$x^2 - 16$** y-intercept: ___________. x-intercept(s): ___________

b. **$y = -x^2 + 8x$** y-intercept: ___________. x-intercept(s): ___________

APPLIED PRACTICE

15. The height, **h**, as a function of time, **t**, that you can throw a ball on the earth is given by:

$$h_{earth} = -5t^2 + 20t + 2$$

where **h** is height in meters and **20** is the speed at which *you* throw the ball straight up (in meters per second). ***t*** is the time in seconds of the ball leaving your hand. The constant **2** is because the thrower releases the ball **2** meters from the ground. Don't worry about the physics of how we know this function.

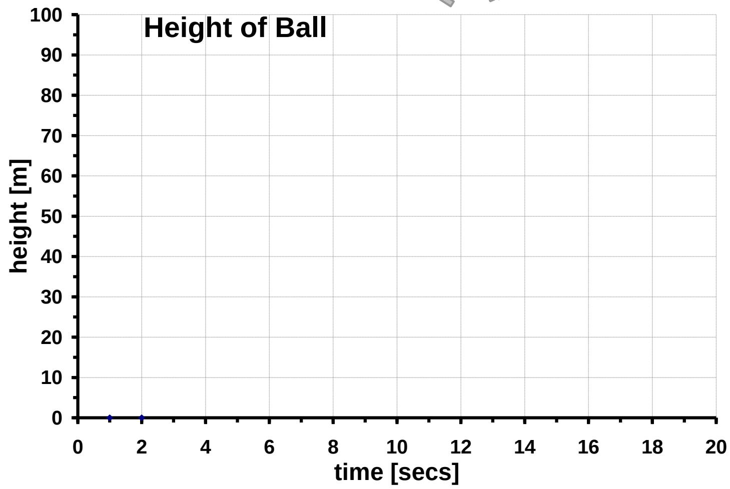
16. On the moon there is much less gravity, so the equation is:

$$h_{moon} = -1t^2 + 20t + 2$$

(again, this is not physics class, do not worry about where I get the formula)

17. Manually graph the two functions (earth and moon) on the graph paper below. Use a Domain scale of 0 to 20 seconds along the x-axis (time). Use a Range scale of 0 to 100 meters on the y-axis (height).

Moon		Earth	
t	h_{moon}	t	h_{earth}
0		0	
5		1	
8		2	
10		3	
12		4	
15		5	
20			



[Workspace:]

a. where is the vertex of each function? Earth and moon?

Vertex Earth: (___ , ___)

Vertex Moon: (___ , ___)

b. what is the equation of the axis of symmetry *or* line of symmetry of each function?

Axis of Symmetry Earth: x = ________ .

Axis of Symmetry Moon: x = ________ .

c. what is the *y*-intercept of each function (ie: when time is zero)?

y-intercept Earth:____________ or (0, __)

y-intercept Moon: ____________ or (0, __)

d. what are the (*approximate*) *x*-intercepts of each function? [Right most when ball hits the ground]

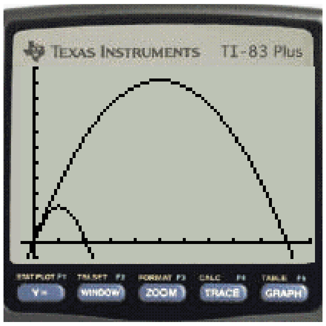
y-intercept Earth:____________ or (___ ,0)

y-intercept Moon: ____________ or (___ ,0)

e. **Solving a quadratic Equation**! Approximately what time does the ball reach **20** meters height:

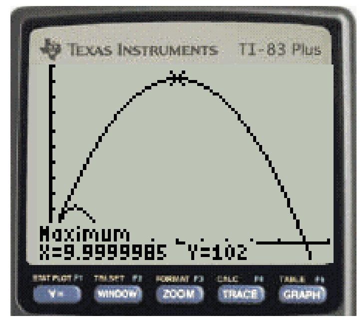
on the earth: ______________ ; on the moon: ______________

Now check all these graphical calculations using the Graphing Calculator. You should end up with some of the pages looking like this:

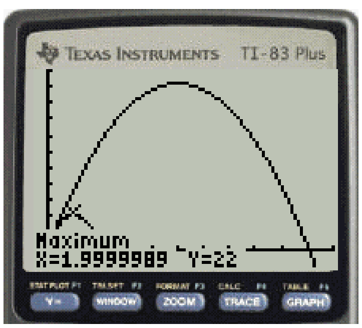


The two curves, tick marks adjusted

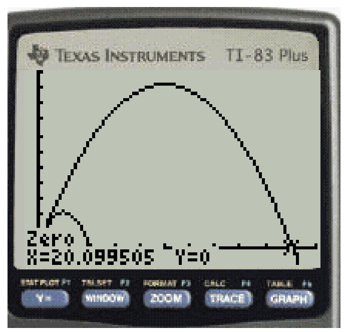
The window size I used



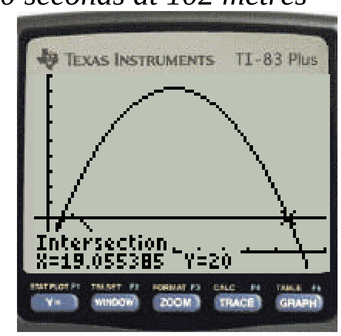
Max height (vertex) moon
10 seconds at 102 metres



Max height [vertex] earth
2 seconds, height of 22 metres



right most x-intercept when height is zero
ball hits ground on moon at 20 seconds



*

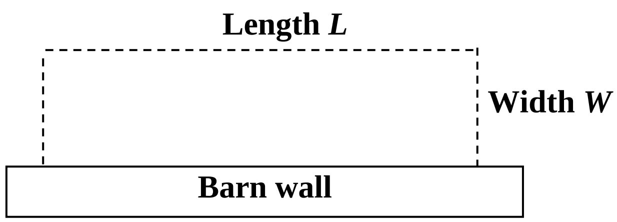
Solving! intersection when the curve equals 20

Of course there are tons of Apps and websites that will do the same type of graphing! Investigate those.

MAXIMIZING AND MINIMIZING STUFF

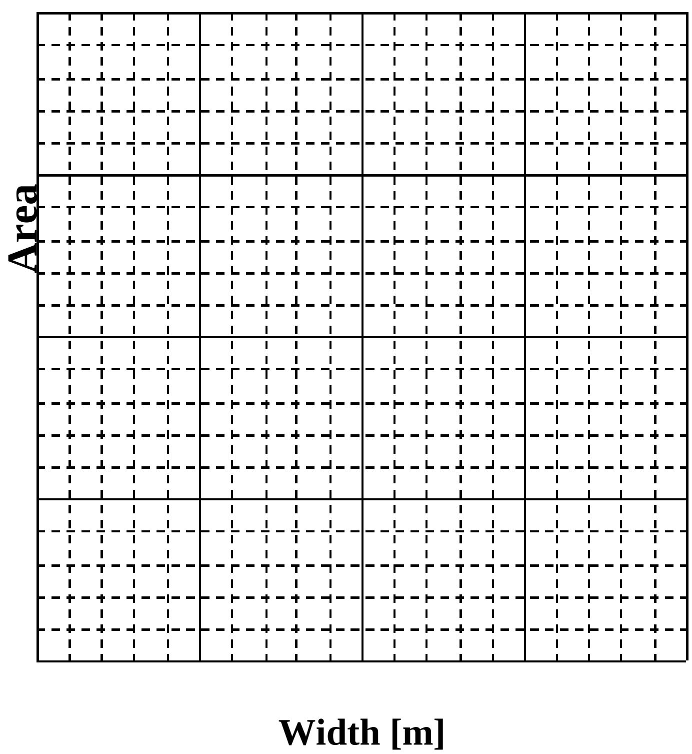
18. You have noticed that quadratics have a **maximum** or **minimum** value that they reach. Knowing when something reaches a peak, or bottoms out, or changes in a better direction is important in math and in life!

19. Walter has an emu farm. He wants to fence off a **maximum** rectangular amount of area against his barn so that his *emus* have a *maximum* area to graze in. Walter has 30 meters of wire fence to make the enclosure.



20. What is the *maximum area* he can fence off with his **30** metres of fence if one side is his barn? What dimensions of length and width will give him the maximum area? Complete the table and manually graph it below.

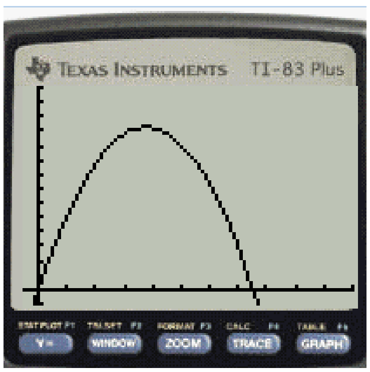
Hint: **30 = 2W+*L*** (so that L = 30 – 2W) and ∴ **Area = L*W**



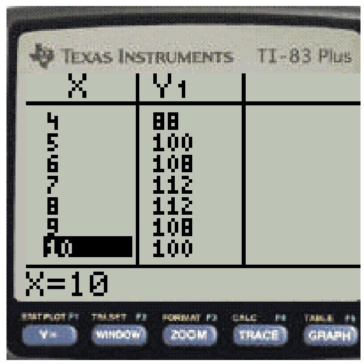
W [m]	L [m]	Area [m^2]

Workspace:

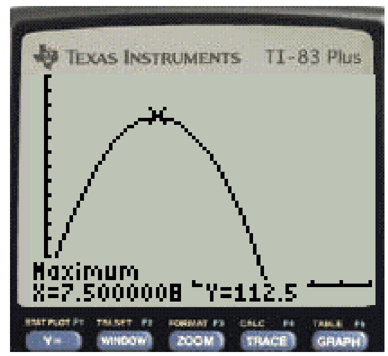
21. Check it all using a **TI-83** or any graphing tool and the formula **y = x*(30 – 2x)** where we are using **x** for the width now. Notice that **y = $-2x^2 + 30x$** is the more familiar polynomial general form when you multiply (**F.O.I.L.**) it out. Your TI-83 should have given results similar to these pictures below



The Graph
For Emu pasture



The Table



The Vertex (Maximum)

APPLIED EXAMPLE OF QUADRATIC– BRAKING DISTANCE!

30. The braking distance of car, once the brakes are applied, is a function of slipperiness of the road *and* speed. In fact, the stopping actually depends on the **square** of the speed! (*Really*! **Double** your speed: **quadruple** your stopping distance! Or quadruple the damage to your head when you hit a telephone pole). The formula for the braking distance then is like this:

$$\mathbf{D = .01*K*}\boldsymbol{v}^{\mathbf{2}}$$

In this formula, **D** is distance to stop in meters. The initial speed when you start braking, ***v***, is in km/hour. The **K** is an extra *coefficient* to allow for slipperiness and friction with the road. When **K** is **1** the road is **normal and dry**. When **K** is more than 1 **(K>1)** the road is slippery. On ice, **K**, can be as high as **5**.

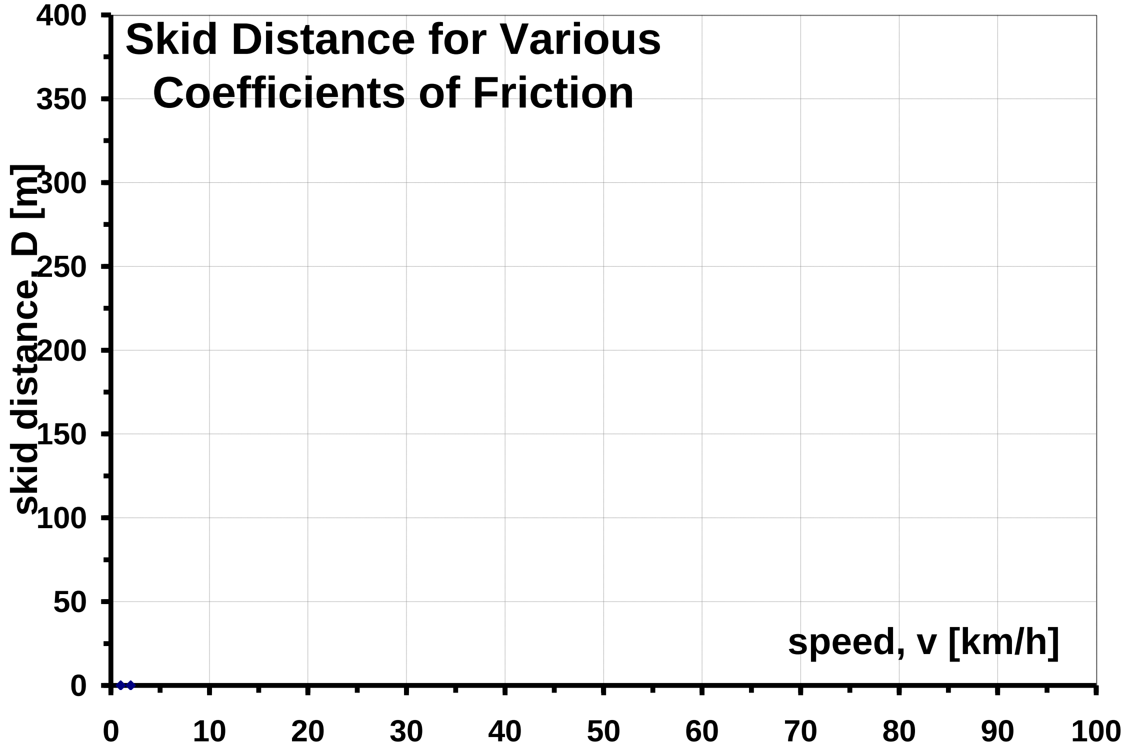
31. What distance does it take for the car to stop if initial speed, **v**, is **50** km/hr and the roads are dry, **K=1**. ?__________________ (hint: *evaluate*, plug in)

32 What distance does it take for the car to stop if initial speed, **v**, is **100** km/hr and the roads are dry, **K = 1**. ?__________________

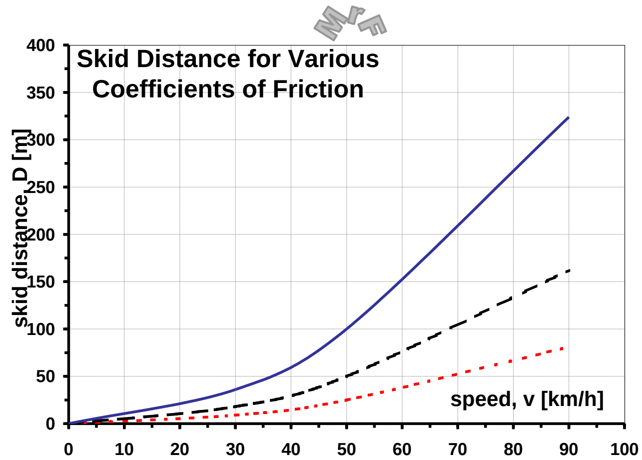
33. Complete the blanks in the stopping distance table:

Stopping Distance, *D*, Table for various Friction coefficients, K and velocities, *v*			
v	**K**	**D = .01*K*v^2**	**Value of D**
30	1	D =	
60	1	D =	
90	1	D =	
30	2	D =	
60	2	D =	
90	2	D =	
30	4	D =	
60	4	D =	
90	4	D =	
??	**4**	D =	**80 meters**
??	**1.75**	D =	**40 meters**

34. Now graph below the data from the table we calculated above!. (You will need to graph three separate parabolic curves, one for each given '**K**').



Your graph should resemble this:



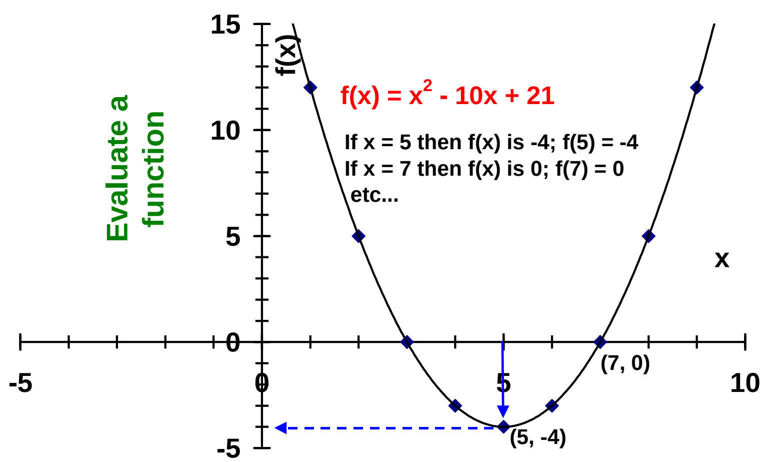
Notice how by doubling your speed you quadruple your stopping distance!

Graph all three curves on the calculator!
Check the table, check the graph.
Play with the TI-83 **Table Setup**
How fast was someone going if their skid mark was 80 meters long on a dry day?

____

SOLVING QUADRATIC EQUATIONS

55. So far we have '*evaluated*' some functions and '*graphed'* what they look like. To evaluate a function means to plug in different values of *x* to see what single value the function has for values for each x. And by graphing them we have seen where the functions have maximums, minimums, and where the function has a value of zero, etc.



56. **Solving.** Involves doing *evaluation backwards*. If you are told what the value of the function is **then what is the x** that gives you that **y-value**. **Example**: above, if you are told the function has a **value of 5** at some place(s) then what value(s) of x make that true?

x = ___________________________ (*notice there are two answers*)

57. **Solving using a graphing tool**. In Applied Math we use a graphing tool to solve functions. (In Pre-Calculus they learn how to solve many types of equations using only algebra). Any graphing tool will give you a *pretty close* answer for a solution to an equation but the TI-83 Graphing calculator has some very simple steps.

58. To solve an **equation** such as $\mathbf{x^2 - 10x + 21 = 5}$ (which means what value(s) of x makes the function have a value of **5**) with a **TI-83** graphing tool simply:

a. graph the function $\mathbf{x^2 - 10x + 21}$ in **Y1=**

b. graph the function $\mathbf{y = 5}$ in **Y2=** (A horizontal line in this simple case)

c. find the point(s) of intersection (where an ***x*** gives the value of 5) using the **2nd** **TRACE** **5:INTERSECT** feature of the TI-83.

d. The screen will prompt you for which two curves you are interested in finding the intersection of. If you only have two functions entered in to **Y=** page then just hit **ENTER** twice.

e. Then move the '*spider*' close to the intersection you are looking for as a solution and hit **ENTER** again. The spider jumps exactly to where they cross and the place where they cross is displayed at the bottom of the screen as an ***x*** and a ***y***.

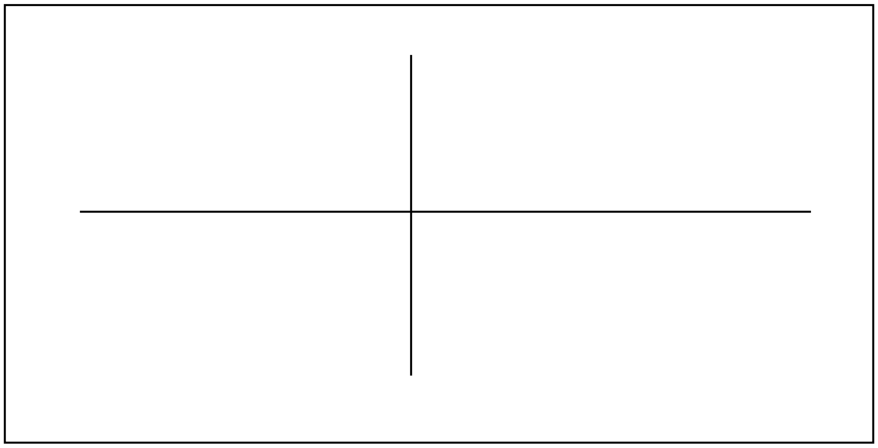
f. notice in this case there are two values of ***x*** that make the function $\mathbf{x^2 - 10x +21 = 5.}$ So you will need to do the intersection finding twice for the two points of intersection.

59. So the solution(s) to**:** $\mathbf{x^2 - 10x +21 = 5}$ **is: x= ____________ and x = _______**

↑ GRADE 11 REVIEW ↑ IS COMPLETED

CUBIC FUNCTIONS

60. A cubic function is one of *degree* three. Eg: $\mathbf{f(x) = x^3 - 4x}$. Graph this non-linear function with the value of the function on the y-axis with a graphing tool and **sketch** its graph below:

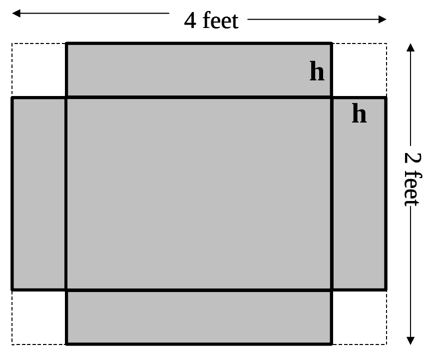


61. Notice that a cubic equation has two '*bumps'* in it (it often, but not always, has a 'local' maximum and a 'local' minimum too). These types of functions are common for finding volumes of Three-Dimensional figures. Recall from Grade 10 a few cubic functions we used such as: $Vol_{sphere} = \frac{4}{3}r^3$ or $Vol_{cube} = (edgelength)^3$

62. You want to make a cylindrical tank under your kitchen sink for your recycling. It can only be **40 cm** high. You want it to hold a **volume** of **0.5 cubic meters**. What radius must you make your tank? Recall the volume of a cylinder is given by:
$V = \pi r^2 h$ where ***h*** is the height and ***r*** is the radius. *Caution: you will need to convert the 40 cm into meters, since you cannot mix different measures of length in a formula*.

Ans: _______________

63. **Advanced Question**. The Nisga'a first Nation in BC loved your excellent Manitoba jingle dancers and granted you a piece of sacred cherry, a board 2 feet by 4 feet to make a 'box' for your jingle dresses. The box will have no lid. You recall that the volume of a box (or more correctly, a rectangular prism) is given by the formula Volume = Length * Width * Height.



a. draw out the '**net**' of the box→.

b. write the formula in terms of height.

Vol = (4 – 2h) * (2 – 2h) * h

c. Graph the function on a Graphing tool to answer the following questions.

d. what is the volume of your box if you make the box 8 inches high?

e. what is the volume of the box if you make box 12 inches high?

f. what is the volume of the box if you make it 6 inches high?

g. what height should you make the box to maximize its volume?

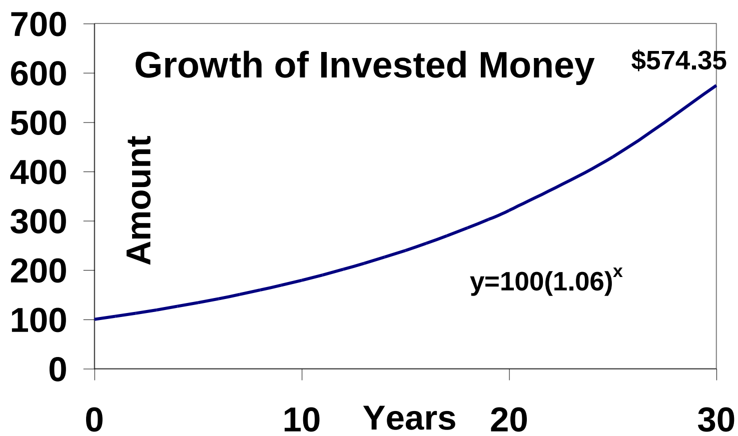
h. What are the value of all three dimensions (length, width, height) of the 'box' to maximize its volume?

i. what height would you make the box to give it a volume of exactly 1.5 cubic feet?

EXPONENTIAL FUNCTIONS

65. A very important function that you experience *every day* is the **exponential function**. It is called exponential because the **variable** is an **exponent**. A basic exponential function has the form: $\mathbf{f(x) = ab^x}$. Exponential functions are ***not*** polynomials.

65. **An example**: The value of **100** dollars left to grow in a bank account at **6%** annual interest can be given by **Value = 100 (1.06)**^{t}. Where t is the time in years.



a. How much money do you have in your account after 10 years? ________

b. When does your money double to $200.00? _____________

66. Mould and germs and diseases also grow exponentially, and the population of the earth grows exponentially (until they run out of food and resources of course, there is a different function for that though!).

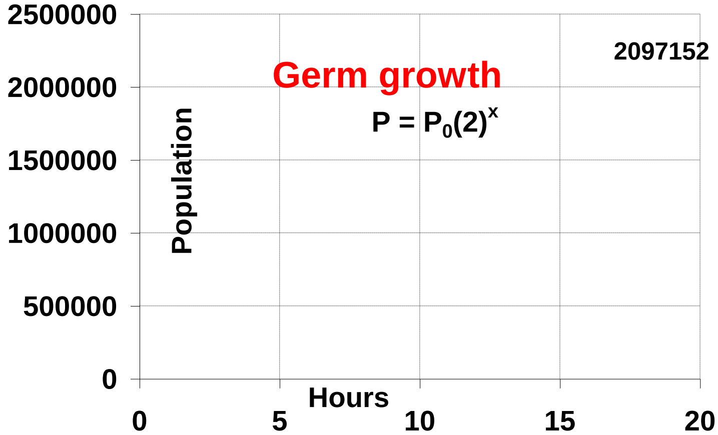
67. **Germ Growth Example**. Those nasty germs and viruses that you hate do grow *exponentially*. Say someone leaves two germs on a doorknob. Germs reproduce just like all living things; and say these germs double themselves every hour! Make a table of their growth:

Hours	0	1	2	3	4	5	6	10	20
# Germs	2	4	8	16	32				

Notice how our elapsed time above starts at time = 0, people often forget to start counting time at zero for some reason! Time of zero is 'now'!

68. Graph the Germ Growth on the next graph given that the equation is given by $\mathbf{P = P_0(2)^t}$; where **P** is the population (number of germs) at any particular time, $\mathbf{P_0}$ is the ***initial*** population (ie: when **t = 0**), and **t** is time in hours.

69. Drawing the exponential curve for the Germ Growth from above:

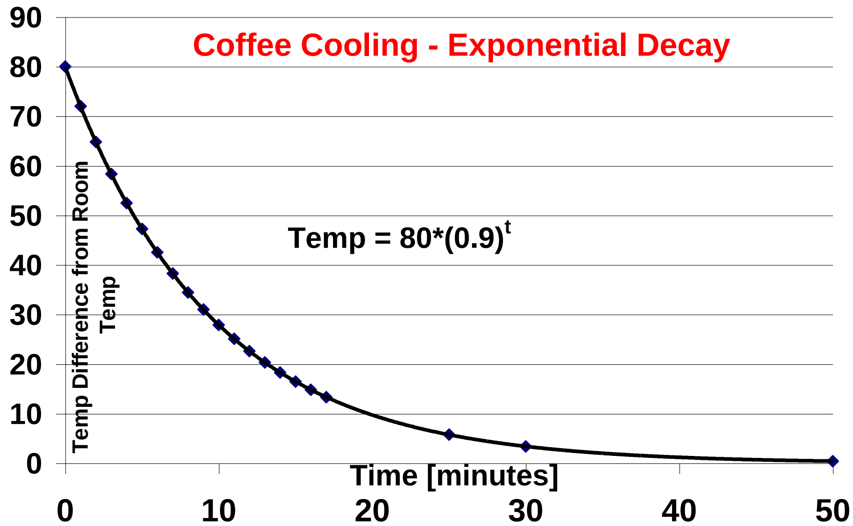


After 6.5 hours, how many germs on the door knob? __________

At what time are there one million germs on the door knob? ______

70. **Exponential decay**. In the exponential form **y = ab^x**, if the **base** is less than one but greater than zero ($0 < b < 1$), then the curve is said to ***decay***. Examples: Radioactivity, objects that cool, a sound dying out, etc are all said to decay ***exponentially*** as a ***function*** of time.

71. **Example Exponential Decay**. Have you ever noticed that your coffee seems to get cooler quicker when it is hot when you first get it? It cools more quickly in the beginning, but towards the end takes longer to cool off to room temperature.



72. **Characteristics of Exponential equations**.

a. **Domain**: all real x. **($-\infty < x < \infty$).** Consequently they have a y-intercept.

b. **Range**: notice the range *tries* to reach a **minimum** value at infinity (or negative infinity) in the x direction. The line that a curve gets *close too but never reaches* (eg: room temperature in the case of our coffee) is called the **Asymptote** line. Sketch in the asymptote in the coffee curve above.

c. **One-to-one**. Exponentials are *one-to-one functions*; they only have **one value** of **y** for each **x**, and each value of **y** has only one **x** that makes it, the y value is ‘unique’. Functions like sine and quadratics and cubics nor any polynomial function in general are not one-to-one.

d. **Half Life**. In *exponential decay* situations things **decay,** not grow**,** by a certain percentage for each change in the **x,** usually time. In the case of our coffee the temperature dropped by **one half in 6.5 minutes** to **40C**, then in **another 6.5 minutes it dropped to half again** *from* **40C** *to* **20C**, then in **another 6.5 minutes (for a total of 19.5 minutes) it dropped half again** *from* **20C** *to* **10C**. The **time** it takes to **decay to one half** of a previous value is called a ‘**half-life**’. You will hear the term **half-life** a lot concerning radioactivity. The half-life of some radioactive substances can be thousands of years! So radioactive substances may be only **half as dangerous** as they were **every 10, 000 years** for example.

73. Calculate the following to 2 decimal places

a. What temperature is your coffee after 15 minutes? ________

b. When does your coffee cool to 10 degrees C? ________

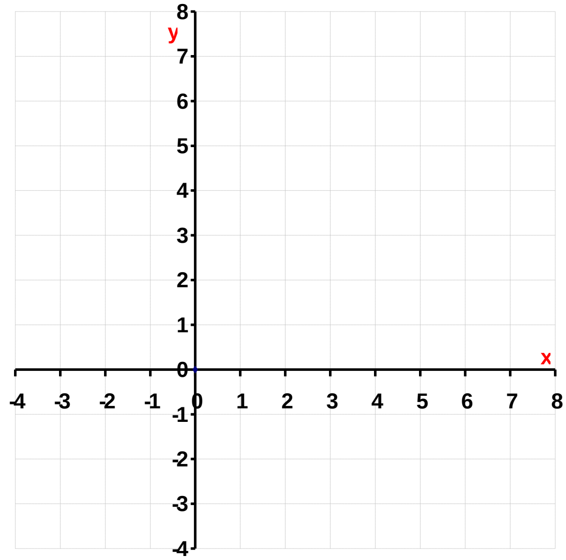
c. Calculate the half life of your coffee’s temperature using a graphing tool:
____

GRAPHING AN EXPONENTIAL FUNCTION - BASICS

74. Manually calculate and graph the exponential function:

$$\mathbf{f(x) = 2^x}$$

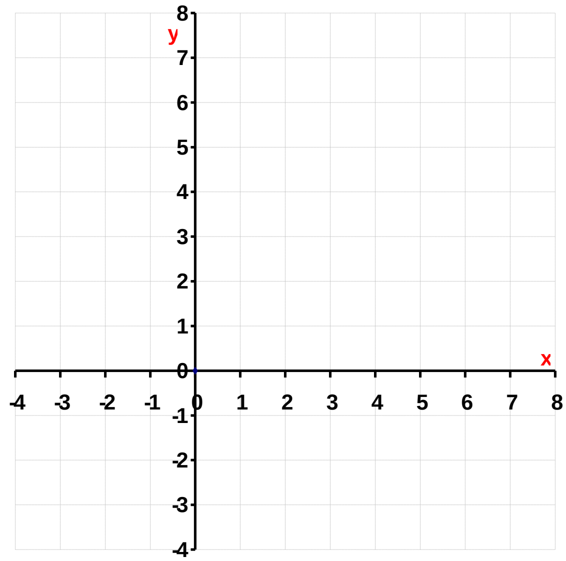
x	**f(x)**
$^{-}$**3**	
$^{-}$**2**	
$^{-}$**1**	
0	
1	
2	
3	



75. Manually calculate and graph the exponential function:

$$\mathbf{f(x)} = \left(\frac{1}{2}\right)^x$$

x	**f(x)**
$^{-}$**3**	
$^{-}$**2**	
$^{-}$**1**	
0	
1	
2	
3	

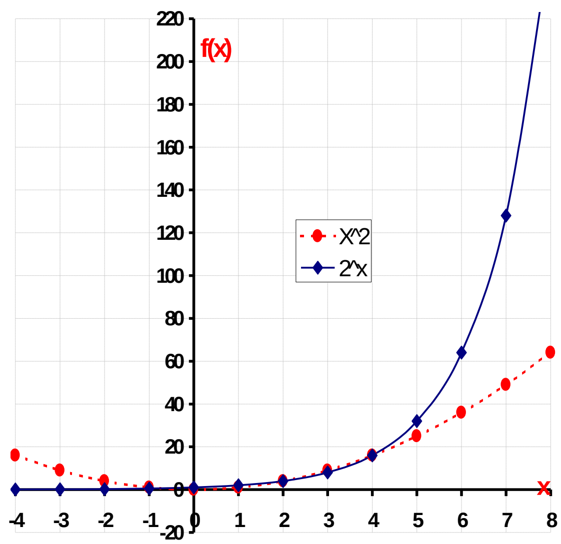


76. Make no mistake, an exponential is not the same as a quadratic! Notably, exponential functions 'take-off' eventually, even if they seem to start out slowly, and easily surpass polynomial functions in their y-values eventually. Complete the table to see how exponential functions grow rapidly after a slow start:

x	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
$\mathbf{x^2}$	**1**	**4**	**9**	**16**	**25**			**64**
$\mathbf{2^x}$	**2**	**4**	**8**					

78. The exponential function is '***one-to-one***'. The y-values are unique. When you solve for **y** there is only one answer. Unlike sine function or quadratic function.

79. The exponential function starts out slow but will readily surpass the power (polynomial) function such as $\mathbf{x^2}$.



LOGARITHMS

80. We use Logarithms to Solve exponential equations without graphing. Let's explore how we solve some of the basic equations we have learned so far.

81. Solve the following equations by using algebra (and check with a graphing tool)

a. $\mathbf{2x + 1 = 7}$ x =

b. $\mathbf{2x + 1 = 3x - 4}$ x =

c. $\mathbf{x^2 = 9}$ x =

tricky; two answers

d. $\sin \theta = 0.5$

tricky; two answers

e. $\sqrt{x-4} = 2$ x =

SOLVING EXPONENTIAL EQUATIONS USING LOGARITHMS

82. So how would you solve this using just algebra?

$$\mathbf{100*(1.05)^t = 200}$$

(an exponential equation)(this is like asking when does my $100 become $200 at 5% annual interest)

83. That is what your **LOG** and **Ln** buttons on your calculator do! They solve exponential equations; it allows you to undo an exponential equation of the form $\mathbf{y = ab^x}$ to isolate the unknown variable **x** all by itself!

84. Try this fancy algebra: if we say we have a function $y = 2^x$ and we want to know when the **y = 32**; in other words we want to solve the equation:

$$\mathbf{32 = 2^x}$$

Enter this in your calculator: $\frac{\log(32)}{\log(2)}$; it had best say the answer is **5**. Of course you should check: is $\mathbf{2^5 = 32}$?

85. Try this from a previous example above in which we graphed the solution:

The value of **100** dollars left to grow in a bank account at **6%** interest can be given by $\mathbf{Value = 100\,(1.06)^t}$. Where ***t*** is the time in years. When does you money double to **$200**.

86. Here are the algebra steps:

- $\mathbf{200 = 100(1.06)^t}$ *The equation to solve*
- $\frac{200}{100} = (1.06)^t$ *Divide both sides by the co-efficient in front of the power to isolate the power.*
- *Simplify:* $2 = (1.06)^t$
- $\mathbf{\log(2) = \log(1.06)^t}$ *Perform the 'log' operation on both sides*
    - $\mathbf{\log(2) = t * \log(1.06)}$ *'Pop' the exponent down in front of the log*
- $\frac{\log(2)}{\log(1.06)} = t$ *Isolate the '**t**', divide by both sides by log (1.06)*
- *Use your calculator:* $\frac{\log(2)}{\log(1.06)} = 11.8957$ years.

Compare to your answer with the graphing calculator above.

87. **Ln Button**. By the way, the **Ln** button could have also been used above!

Solve the following exponential equations (show work!): Check with a graphing tool.

a. **$4{,}000 = 1{,}000 * 1.08^x$**	**Ans: x = 18.01**
b. **$20 = 90 * (0.93)^t$**	**Ans: t = 20.73**
c. **$512 = 2 * 2^x$**	**Ans: x = 8**

COMPARE LOGARITHM TO OTHER FUNCTIONS AND THEIR INVERSES

88. The logarithm function '*undoes*' the exponent function. You have been 'undoing' lots of functions ever since Grade 8. A function that *undoes* another function is called its ***inverse function***. Let's review a couple inverse functions and then the logarithm function.

Dividing undoes multiply

x→	**2 * x**
1	**2**
2	**4**
3	
	8
	12
x ÷ 2	**←x**

Complete the blanks

Square root undoes square

x→	**x^2**
2	**4**
3	**9**
10	
	16
	81
$\sqrt{x}$	**←x**

Complete the blanks
Going right is 'squaring';
going left is 'square rooting'

Logarithm *undoes* exponent

x→	**10^x**
0	**1**
1	**10**
2	**100**
	1000
	10,000
$\log_{10}$ x	**←x**

Complete the blanks
A log undoes an exponent.
This is a base of 10; other bases are possible.

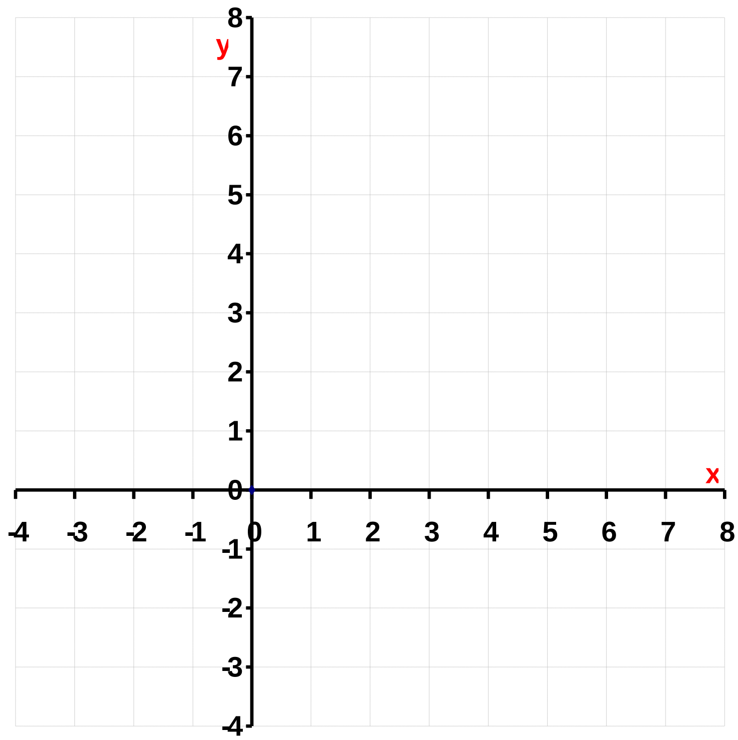
GRAPHING THE LOGARITHM FUNCTION

89. Since the Logarithm function just undoes the exponential function its graph should likely look somewhat similar?

90. Complete the table and graph and label the base 2 Exponential function *and* the base 2 Logarithm function.

$x \rightarrow$	2^x
-3	
-2	$1/4$
-1	
0	1
	2
2	4
	8
$\log_2 x$	$\leftarrow x$

Recall that $b^{-m} = \dfrac{1}{b^{+m}}$?



$\mathbf{LOG_{10}}$ vs Ln = COMMON BASE 10 LOGARITHM vs NATURAL LOGARITHM

92. Your calculator has two LOG buttons. The LOG button itself is for base 10. Anytime you see just LOG it implies that it is for a base of 10 : **$\mathbf{Log_{10}}$**.

A real mathematician or scientist would not use Base 10, they would use, ….

get ready for it, ……

‘**e**’.

WHAT IS THE NUMBER 'e'?

94. To make things simple think of '**e**' as being magic, sort of like **π** is magic. '**e**' is most definitely a number; its approximate value is: **2.718281828459045235360287471352**7…
and like **π** it is an *irrational number* that cannot be written as a fraction nor consequently as a decimal.

You will have to trust that '***e***' really is magic, hang out with your teacher if you really want to know why.

95. **Ln is Log$_e$**. The **N**atural **L**ogarithm, **ln**, has '**e**' as its base. Occasionally you will see **Ln** written as **Log$_e$**. For the purpose of this course the **Ln** button is sufficient for solving equations.

96. **Example - Solve Exponential Equation with ln**. The number of germs on a door knob grow *continuously* as a function of time in accordance with the exponential expression $\mathbf{N(t) = N_o * e^{t/4}}$. The key to knowing when the natural number '**e**' is being used is when the problem talks about '*continuous* growth'. So in this example **N(t)** is the number of bacteria (whole numbers) as a function of time ***t***, $\mathbf{N_0}$ [pronounced 'N naught'] is the initial number at time zero (naught in English), and **t** is time measured in hours.

97. So if we start with $\mathbf{N_0}$ = two germs then $\mathbf{N = 2e^{t/4}}$
Let's find when we have a thousand germs on the door knob

$\mathbf{1{,}000 = 2e^{t/4}}$

$\mathbf{500 = e^{t/4}}$ Divide both sides by 2 to isolate the power

$\mathbf{\ln(500) = \ln(e^{t/4})}$ Perform Natural Log of both sides

$\mathbf{\ln(500)} = \frac{t}{4}\ln(e)$ Pop the exponent down in front of the ln

$$\frac{\ln(500)}{\ln(e)} = \frac{t}{4} \quad \therefore \quad \frac{4*\ln(500)}{\ln(e)} = t$$

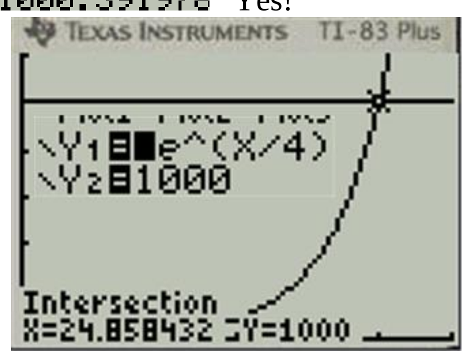
Isolate the t

t = 24.86 hours

Solution: 24.86 hours to get 1,000 germs

Check answer!! $\mathbf{1{,}000 = 2e^{24.86/4}}$ **???** 

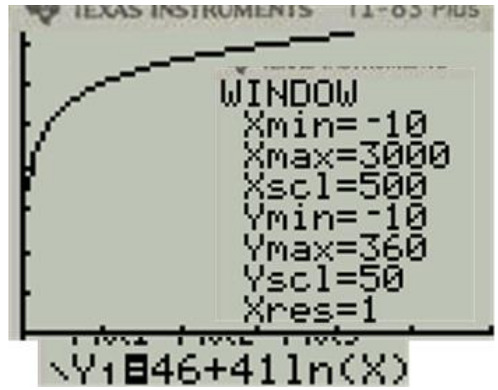
Yes!



98. Alternately the problem could readily be solved by graphing!

Technically speaking Applied Math students need only know how to do graphical solutions and with the aid of graphing tools.

99. **Example - Solve a Ln Equation**. The altitude of an airplane is measured in 100's of feet in what are called **Flight Levels** (FL). Thus if an aircraft is at **FL350** it is at **350** ***hundreds*** of feet, so **35,000** feet in altitude. Aircraft tend to climb to the highest optimum cruise altitude and of course as they burn off fuel weight they can *slowly* climb higher. The cruise altitude, **A**, in units of **FL** of a particular aircraft is a function of the distance, **d**, in miles it flys and is modelled by **A = 46 + 41ln(d)**.



a. Graph the logarithmic curve of the height vs distance to get this→

b. Evaluate the cruise level formula to calculate at what altitude the aircraft should be after 2500 miles. **A =** _____________

b. using a graphing tool calculate at what altitude, A, the aircraft should be after 2500 miles. **A =** _______________

c. solve to find the distance flown if the aircraft is at FL330. **d =** ___________

APPLICATION OF LOGARITHMS

100. We use LOGS when we talk about earthquakes (The Richter Scale), when we do Chemistry (the $\mathbf{P_h}$ or strength of an acid), when we measure volumes of sounds (Decibels), or brightness of stars or calculate the age of an ancient artefact using carbon dating. In all these cases we don't talk about the amount of destruction or volume or acidic concentration strength, we talk about the *exponent of the amount* of those, otherwise the actual number would be way to big or way to small to write and would become very cumbersome.

101. **Sound Volume – Decibels**. Invented by Alexander Graham Bell, an honorary Mohawk! Since sound can be so very quiet or so very loud he found it convenient just to talk about the logarithm (the exponent) of the sound level. A cat purring at one metre away has a sound level of about 25 decibels. This would be the exponent of the sound power reaching your eardrum in watts per square cm on your ear drum (or something like that which is not important here). To slightly complicate things, Alexander Graham Bell did not like numbers that had decimals in them, so when he said 25 decibels (dB), he meant 25 tenths (a *deci*-) of a Bel; so 2.5 Bels.

102. Regardless of the technicalities, a sound that is 20 decibels louder than another sound is actually doing 100 times as much damage to your ear! Since 20 tenths of a Bel , is 2 Bels which is an exponent of 2 on a base 10 Log and $\mathbf{10^2 = 100}$.

103. Most doctors will say that prolonged exposure to an 85 dB sound will quickly cause ear damage. How much more damage are you doing if you have your music cranked up to 100dB?

104. $Power\ increase = 15db$; = 1.5 bels. But a logarithm is an exponent.
Power increase = $\mathbf{10^{1.5}}$ = **31.6 *times as loud***!

105. The same ideas apply to earthquakes and the Richter scale of an earthquake strength . An earthquake of magnitude **7.6** is how much more deadly than an earthquake of magnitude **6.3**?
$\mathbf{10^{(7.6-6.3)} = 10^{1.3} = 20}$ times stronger!

105. Star brightness (magnitude) is measured in logarithms too! Except **not** with a **base of 10**! Star brightness is measure in logs whose base is **2.5**! Not only that but the intensity is arranged so that big brightness numbers are dimmer. So a star with an intensity of 4.2 is brighter than a star of magnitude 6.8 by a brightness amount of 2.6 magnitudes which is really in an absolute brightness sense $2.5^{2.6}$ times brighter or about 11 times brighter to your eye.

106. So you can explore those types of measurements on your own, but you see that we use logarithms almost everyday of our lives.

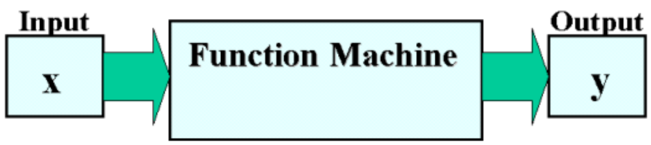
APPENDIX A
TO GRADE 12 UNIT C NOTES
FUNCTIONS

HASTY REVIEW (INTRODUCTION) OF FUNCTIONS

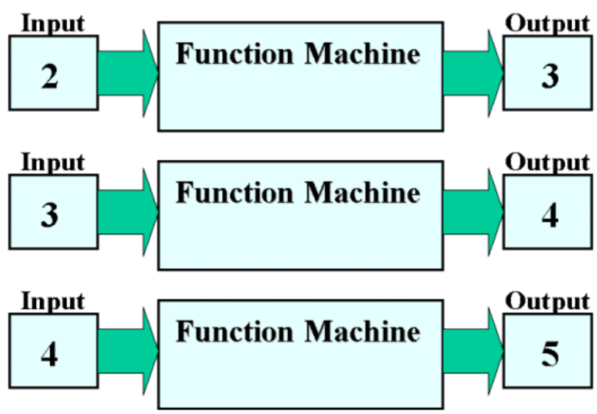
This material should have been covered in Grade 10 and again in Grade 11!

FUNCTION MODELS

1. A function model is the final way to think about the more common relationships between numbers. Using function models we think about a machine that takes an **input** number, **x**, and spits out an **output** number, **y**. It really is as simple as the diagram below:



2. Can you guess what this machine does to an input **x**? Let's call it the **F** machine for short.



Answer:

(Ans: it adds one to the input. f(x) = x + 1.

3. ***f(x)* Notation**. The 'F' machine above took the input of 4 and did a function on it to output the number 5. The process could be written as f(4) = 5 because all the machine does is add 1 to the input number. It is pronounced 'f at 4', or ' f of 4 ' is 5. In the case above f(5) = 6. So what is **f(8)**? Answer: ____________. How did you calculate **f(8)** ? __________________.

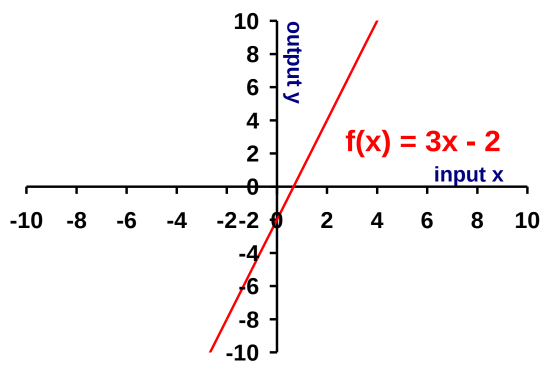
Be very certain: **f(x)** ***does not mean*** some unknown **f** multiplied by some unknown **x**. It just means there is a function that does something to an input of **x**. In fact, it can be **G** machine if you want; ***g(x)*** or whatever you want to call it.

4. Here are several more functions to **evaluate**:

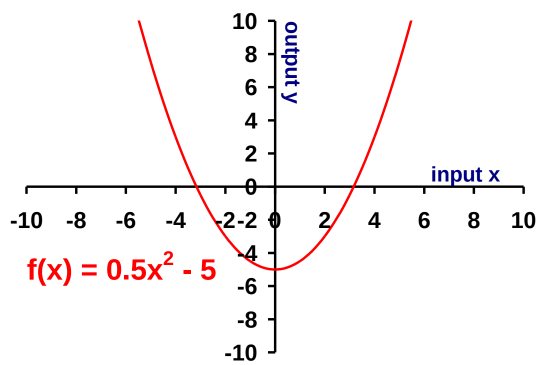
Input: x	Function Process	Output:f(x)
2	**f(x) = 3x + 1**	
½	**f(x) = 3x + 1**	
–3	**f(x) = 3x + 1**	
2.5	**f(x) = x^2 – 4**	
$\sqrt{4}$	**f(x) = x^2 – 4**	
0	**f(x) = x^2 + 2x – 5**	
2	**f(x) = x^2 + 2x – 5**	
7	**f(x) = 4**	
–2	**f(x) = 4**	

Recall from Grade 9; 'evaluate' means to plug in values for unknowns to find a value for the entire mathematical expression

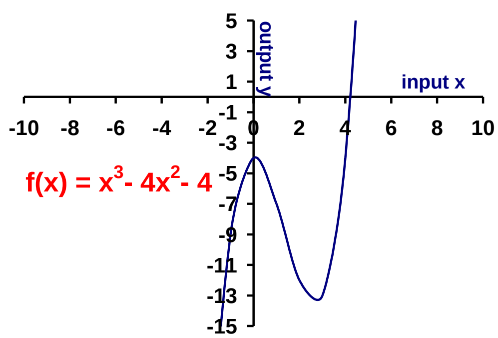
5. Normally we plot the ***input*** of a function on the ***x-axis*** and the ***output*** of a function on the ***y-axis*** of a graph. Below are some typical examples of the graphs of some functions. You can check them with your graphing tool if you want.



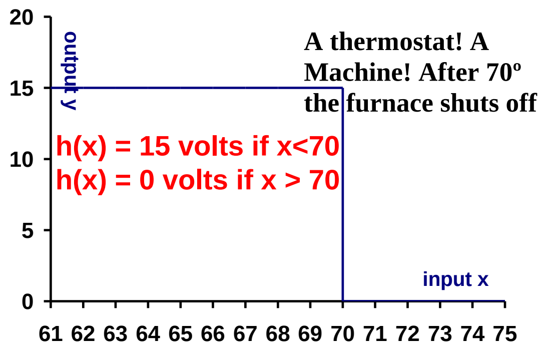
a. Function Graph A



b. Function Graph B



c. Function Graph C



d. Function Graph D. (does it matter if I call the process 'h' for heat instead of 'f'?)

6. **Domain and Range of Functions**. The Domain and Range of a function are often fairly simple and are determinable from the function expression and from the graph of the function.

7. **Domain of a Function**. Normally at this grade level the domain of function is all real values of ' **x** '. So for example, in the four functions and graphs above we see that there is nothing to prevent us from inputting **any value** of ' **x** ' that we want into the machine, it will spit out an answer! So the domain for all of the above function graphs is just: $-\infty < x < \infty$.

8. **Range of a Function**. Often the range of a function **f(x)**, its output being graphed on the y-axis, is limited in some manner though. The range of the four Function Graphs above are:

Function Graph A: $-\infty < \mathbf{f(x)} < \infty$.

Function Graph B: $\mathbf{-5} \le \mathbf{f(x)} < \infty$

Function Graph C: $-\infty < \mathbf{f(x)} < \infty$

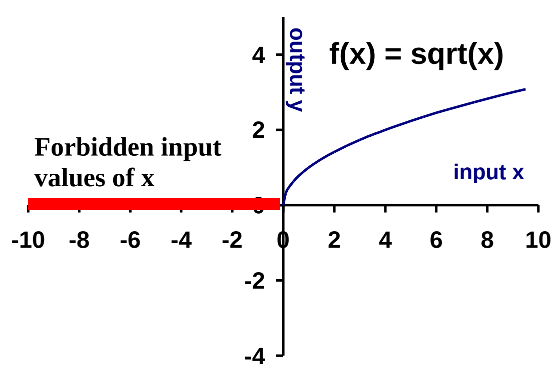
Function Graph D: $\mathbf{0 \le h(x) \le 15}$

9. **A Function with a Limited Domain**. For now, the only function you are familiar with that has a limited domain is the 'square root' function. The square root function would be written in function notation as:
$f(x) = \sqrt{x}$.

10. Are you 'allowed' to do the square root of any number? No! You cannot do the square root of a negative number! So the domain of $f(x) = \sqrt{x}$ is:

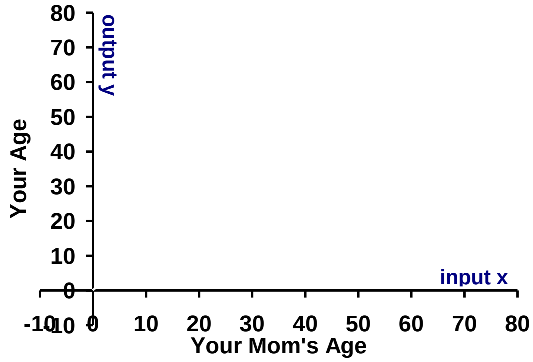
$$\mathbf{0 \le x < \infty.}$$

11. As long as you input a number of zero ***or more***, this function machine will output a number. Anything less that zero and the 'machine' will blow up! (not really!)



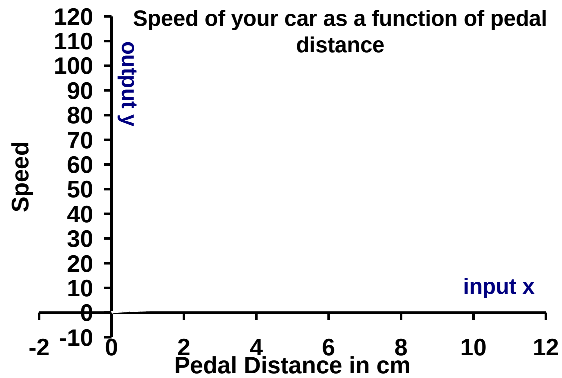
12. Sketch your age as a function of your mom's age.

What are the approximate Domain and Range of this functional relationship?



13. Sketch what the speed of your car as **a function of** how far you push down the gas pedal.

What is the domain and range of the function that models this?



Can you go backwards by pulling up on the gas pedal??

So that is a quick introduction and or review of the idea of functions. Check out the Grade 10 resources if you want more.

APPENDIX B
TO GRADE 12 APPLIED FUNCTIONS
REGRESSION OF FUNCTIONS

REGRESSION

1. **Regression** is a statistical method to find the equation (or a close equation) that expresses the relationship between two sets of values (like coffee temperature and time, or stopping distance and speed, or marks and attendance). Regressions can be done in EXCEL (using a *trend line* of data) or on a Graphing Calculator.

2. Let's do a regression on the following simple data:

x (L_1)	**-4**	**-2**	**0**	**1**	**3**
y (L_2)	**16**	**4**	**0**	**1**	**9**

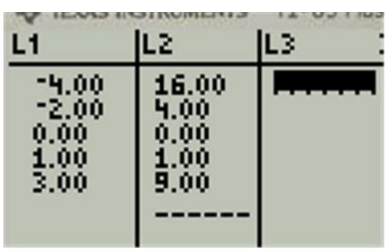
3. **Plotting Data First**.

a. De-select all **Y=** formulas or clear them so they will not graph. The equals sign will be highlighted if they *are* to be graphed.

b. Go **STAT** **EDIT** and enter your data into list 1 (**L1**) and list 2 (**L2**). x's in **L1**, and y's in **L2**.

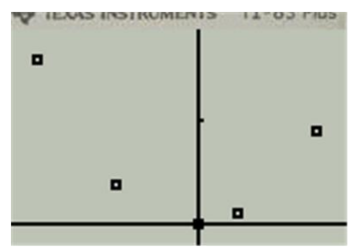
c. Select **STAT PLOT** by pressing **2nd** **Y=**. Select **Plot 1**. Turn on Plot 1 by highlighting ON . Put plot into the *Scatter Plot* type. Make sure the data is being taken from lists **L1** and **L2** for the x's and the y's. Select the largest *mark* possible.

d. Press **GRAPH**. You should have a *plot* of your data! You will need to use **ZOOM 9:ZOOMSTAT** to fit the data just easily and perfectly.



StatEdit Data Table

Stat Plot Selection


Scatter Plot (ZOOM 9!)

4. **Data Regression** [1]. Now that you have your data entered let the TI-83 calculate and graph the *equation* of the regression curve. This is a statistical operation, fitting the best line to the data. Do it like this:

- Go to Catalogue [**2nd** **0**]and select **DiagnosticOn** [2]. Press **ENTER** to paste it to the home screen. Press **ENTER** again to execute the command.

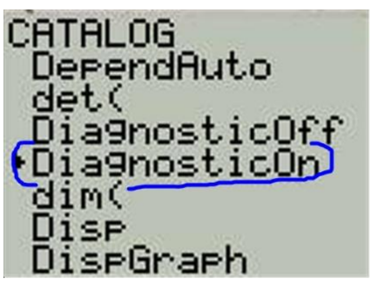
- Press **STAT** . Select **CALC**. Select desired regression (in this case: QuadReg since it certainly plots like a parabola). Press **ENTER**

- The screen will show you the Quadratic Equation that best matches the data. Make sure it makes sense! Make sure the R^2 is close to a value of 1.

- Go to the **Y=** window. Put the cursor in the first function (**Y1=**). Press **VARS** . Select **5:STATISTICS**. Select **EQ**. Select **RegEQ**

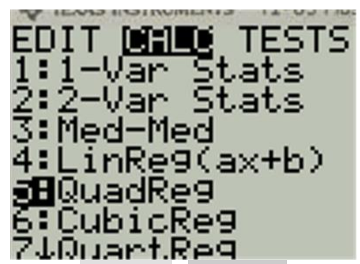
[An alternate way to paste the Regression Equation into the Equation Editor at **Y1=** is to simply paste **Y**$_1$ *after* QuadReg entry is pasted to the screen]

- Press **GRAPH**. Both your raw data plot and the curve of best fit will appear and they should nicely connect almost perfectly.

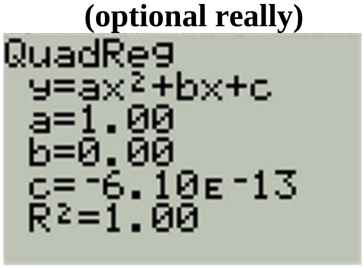


Turn on Diagnostic (optional really)

Turn on Diagnostic

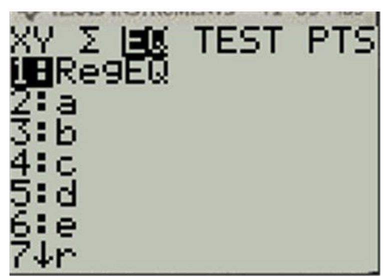


STAT CALC QuadReg



The Quadratic Regression Equation (*I put my MODE to two decimal places to make equation look nicer*)

Selecting the Equation 1



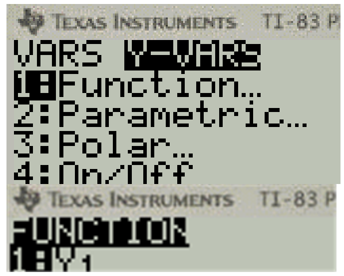
Selecting the Equation 2

---

[1] The **TI 83 Regression** operation is very powerful. There are additional 'parameters' that can be entered. We are just using the most basic and simple mode of **Regression** above.

[2] Turning on the diagnostic will enable the calculator to display the correlation factors!. If r^2 is close to 1 then the equation is a good fit for the data.

Regression Equation pasted into Y =



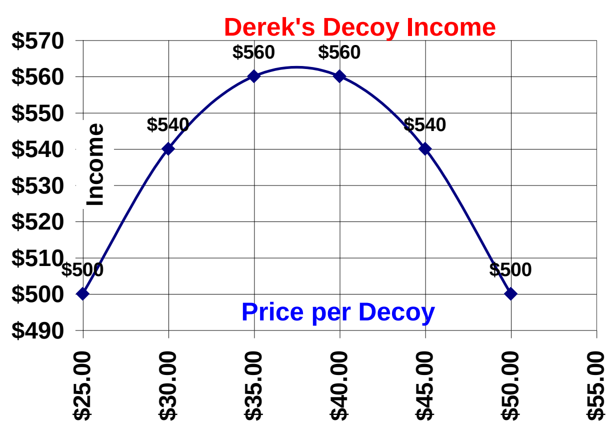
Alternate way to ..

…Paste the equation into the Y =

5. **Example Regression**. Derek makes duck decoys every duck season! They are rather good! He sells them for **$25.00 each**! But he wonders if he should be charging more! When he charges **$25.00** he gets **20** customers every season. But he asked some questions and has determined that if he were to charge **$5.00 more** he will **lose two customers**. And he has determined that every **increase of $5.00** will lose him two further customers with each increase. He has been told by his Math teacher and a Business Commerce graduate that the relationship between selling price and income is quadratic. So what should Derek charge for each of his decoys to maximize his income?

Price	$25	$30	$35	$40	$45	$50
Customers	20	18	16			
Income	$500	$540				

6. **Plot** the data and use a quadratic regression on a suitable graphing tool to find the equation that relates Derek's **income** to the **price** he charges per duck. Your graph should look like this:



7. What is the function or formula that connects all the points (you need to enter at least three points for a quadratic)? ______________________

8. What is the price Derek should charge per duck to make maximum profit and what will that income be? (*ie*: where is the vertex) ________________

(You might be familiar with how to do this sort of stuff in EXCEL also)

REGRESSION OF AN EXPONENTIAL FUNCTION (Decay)
COFFEE COOLING

9. You are observing the temperature of your coffee as it cools; you collect the following data:

Time ,**t** [min]	0	5	10	15	20	25
Temp above room temp, **τ** [°C]	75.0	49.5	32.6	21.5	14.2	9.3

** Get used to weird symbols like Tau '**τ**'; we don't always use the English alphabet!**

10. When does the coffee temperature get down to a Tau, **τ,** of 20°C (above room temperature)

Ans: At 15.87 minutes the coffee is 20°C above room temperature.