GRADE 11 ESSENTIAL
UNIT G - TRIGONOMETRY

INTRODUCTION

1. Trigonometry is the study of three-cornered polygons; *aka*: **tri-angles**. Once you understand the triangle you readily understand every shape (well at least every flat shape) since all other shapes can be made of triangles. Much of the middle half of this unit is a repeat of Grade 10 Essential!

some of this unit (the Sine and cosine Law) is actually Grade 12 Essential curriculum so as to not be rushed in the Grade 12 Course)

ANGLES AND PARALLEL LINES

2. Triangles are polygons composed of three [**straight**] *line segments* forming three corner *angles*. A thorough understanding of lines and angles is a good foundation to geometry and to trigonometry.

3. Examine the diagram below, which may be familiar from your former geometry studies

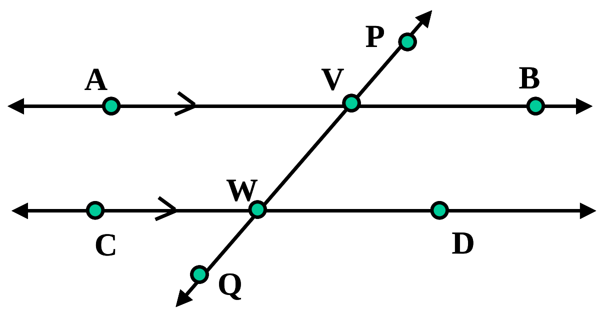


Figure 1

4. Note that $\overleftrightarrow{AB}$ is || to $\overleftrightarrow{CD}$ as indicated by the corresponding '>' carrots or 'chevrons'. The symbol '||' means *parallel*. $\overleftrightarrow{PQ}$ is a line that cuts the two other lines at points **V** and **W** and is called a *transversal*. This particular transversal $\overleftrightarrow{PQ}$ is cutting the two parallel lines.

SUPPLEMENTARY ANGLES

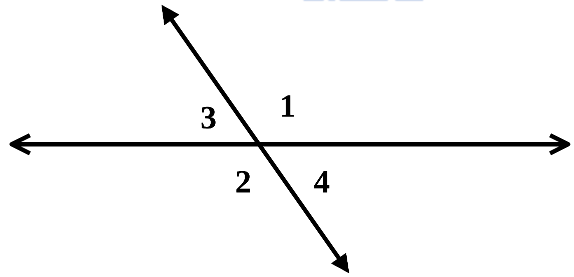
5. You might recall that a line can be considered as two rays butted together at a 180° angle.

Consequently; **∠AVP** is *supplementary* to **∠AVW**, in other words **∠AVP** + **∠AVW = 180°.** Or using some fancy algebra: **∠AVP** = **180°** – **∠AVW**. We say that the two angles *supplement* each other, just like you might supplement your diet. Two angles that are adjacent to each other and make a 180° angle are also called **Linear Pairs**.

OPPOSITE [VERTICAL] ANGLES

6. Have you ever noticed that with angles formed by two intersecting lines the angles across from each other are '*congruent*'; ie: their angle measure is equal.

7. Angles that are across from each other at the intersection of two lines are called '*Opposite Angles*' or sometimes '*Vertical Angles*'.

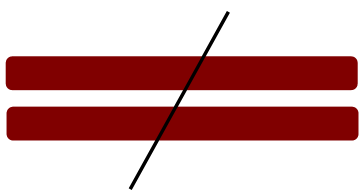


$\angle 1 \cong \angle 2$ *and* $\angle 3 \cong \angle 4$

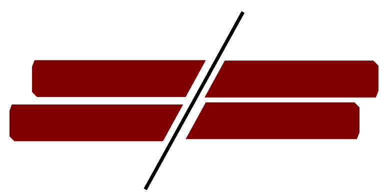
8. For Figure 1 above: Line $\overleftrightarrow{AB}$ intersects line $\overleftrightarrow{PQ}$ at a point V. Consequently, the angle $\angle AVP$ is congruent to $\angle BVQ$. The secret symbol for congruence is '$\cong$' so: $\angle AVP \cong \angle BVQ$

CORRESPONDING ANGLES ON THE OTHER LINE

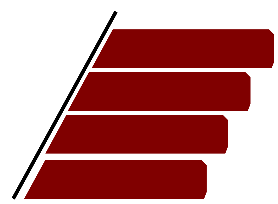
9. Ever noticed than when you cut your kids' wieners to toss in the beans that slicing diagonally across two parallel wieners the angles formed on each weenie are the same? Amazing!!



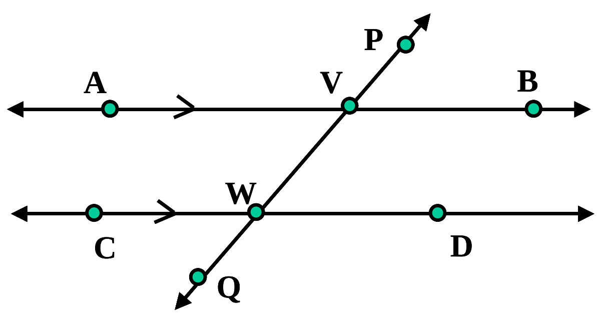
Weiners 1



Weiners 2



Weiners 3



10. Consequently $\angle AVP$ *corresponds* to (matches to) $\angle CWP$ and they are congruent. You could just slide the corner formed downward and they would be the same.

Furthermore, by deductive logic, $\angle BVW$ is *congruent* with $\angle DWQ$.

It is probably best to watch all these rules and deductions in some animated form. So, check out some videos the teacher selects for you. Of course, if you want to see who first explored all this stuff 2500 years ago look up '**Euclid**'. We still have some of his writings!

OTHER ANGLE DEFINITIONS

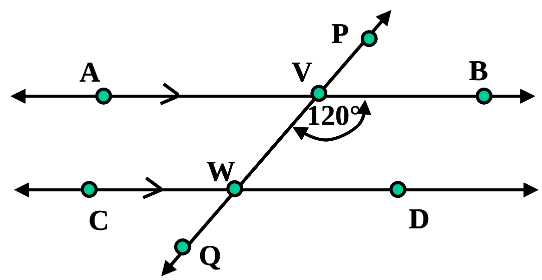
11. Some other angle definitions and laws include.

12. **Alternate Interior Angles**. Angles $\angle BVQ$and $\angle CWP$are a pair of angles on *alternate* sides of the transversal and are *interior* to the parallel lines. They are called Alternate Interior Angles. This particular transversal cuts two parallel lines and we have already observed Alternate Interior Angles of a transversal crossing two parallel lines (or wieners) are congruent.

13. **Alternate Exterior Angles**. You likely already deduced that Alternate *Exterior* Angles, angles that are *exterior* to the parallel lines and on alternate sides of the transversal are also congruent. So for example: $\angle DWQ \cong \angle AVP$

14. Let us apply all these rules to our transversal and parallel lines below.

If you are told that $m\angle BVW = 120°$ then what is the measure of:



$\angle \boldsymbol{AVP} =$_______° $\angle \boldsymbol{AVW} =$_______°

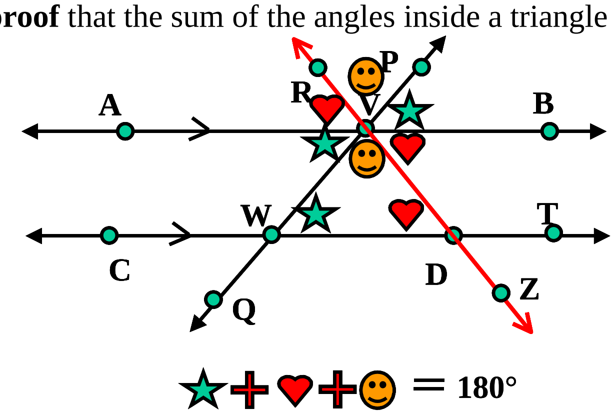
$\angle BVP =$_______° $\angle CWP =$_______°

$\angle \boldsymbol{DWQ} =$_______° $\angle \boldsymbol{DWV} =$_______°

$\angle CWQ =$_______°

THE SUM OF THE INTERIOR ANGLES OF A TRIANGLE ADD TO 180°

15. How is this for a **proof** that the sum of the angles inside a triangle is 180°?

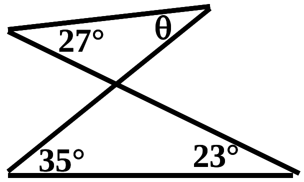


You seldom see proofs of things! You just believe whatever the teacher says do you???

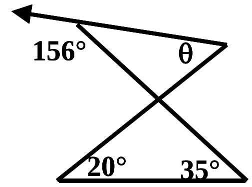
You should **often** say: "***prove what you are saying is true, show me the facts..!***" instead of believing what you are told. Not just in mathematics but in life too! That is why we say that mathematics and science is the **search for truth**.

16. **Example Questions:**

Find angle ‘θ’



Find angle ‘θ’



TRIANGLE CONGRUENCE

17. Knowing when two triangles are congruent that is: the same shape and the same size; is important. There are a couple basic and intuitive laws that I present below without proof.

SIDE – SIDE – SIDE [SSS]
TRIANGLE CONGRUENCE

18. Two triangles are said to be congruent if all three sides in one triangle are congruent to the corresponding sides in the other.

In Figure 2 we say that:

ΔABC ≅ ΔFGH

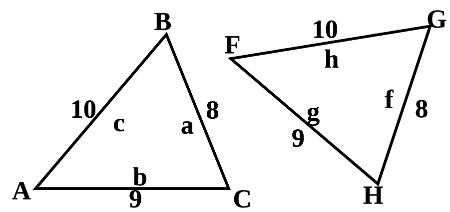


Figure 2

19. The two triangles are congruent since the corresponding pairs of sides are congruent: **a = f; b = g;** and **c = h**.

SIDE – **ANGLE** – **SIDE [SAS]**
TRIANGLE CONGRUENCE

20. Triangles are congruent if any pair of corresponding sides and their ***included*** angles are equal in both triangles

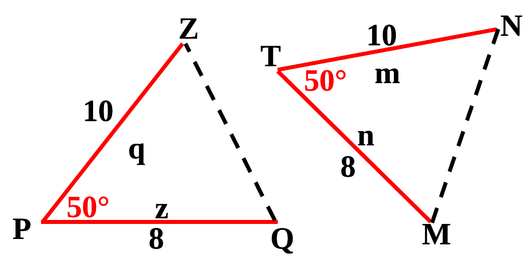


Figure 3

21. The two triangles are congruent, **ΔPQZ ≅ ΔTMN,** since the corresponding pairs of sides are congruent and the angle included between those sides are congruent:
q = m ; **z = n** ; and **∠P ≅ ∠T.**

ANGLE – SIDE – ANGLE [ASA]
TRIANGLE CONGRUENCE

22. Triangles are congruent if any pair of corresponding angles and their ***included*** sides are equal in both triangles.

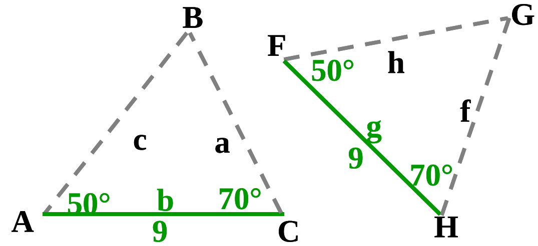


Figure 4

23. The two triangles are congruent, **ΔABC ≅ ΔFGH,** since the corresponding angle pairs are congruent, and the side included between those angles are congruent:
∠A ≅ ∠F; ∠C ≅ ∠H; and b ≅ g

ANGLE – ANGLE – SIDE [AAS]
TRIANGLE CONGRUENCE

24. Triangles are congruent if two pairs of corresponding angles and a pair of opposite sides are equal in both triangles.

By opposite we mean opposite to either given angle, in other words a non-included side.

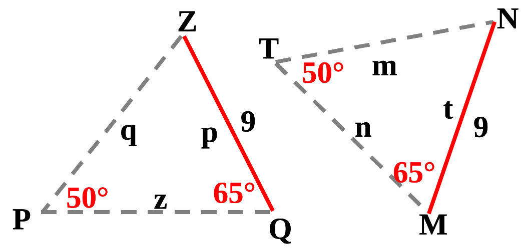


Figure 5

25. **ΔPQZ ≅ ΔTMN** since corresponding angles **∠P≅∠T**; **∠Q≅∠M**; and the non-included sides **p** and **t** are congruent.

(*Can you see from logic that this **AAS** law is really a bit **pedantic**? It really is just the **ASA** law combined with triangle sum of corners = 180° law!*)

ALTERNATE WAY OF SHOWING CORRESPONDING ANGLES AND SIDES

26. Sometimes to show *what* matches (corresponds) *with what* we use tick marks and symbols as indicated at right. The sides are not necessarily congruent however.

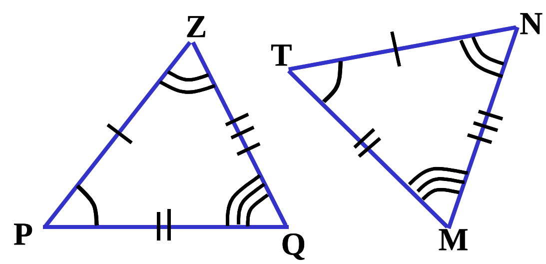
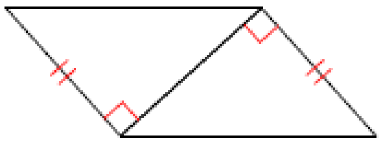


Figure 6

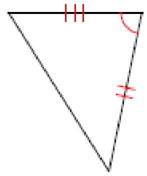
CONGRUENCE EXAMPLES

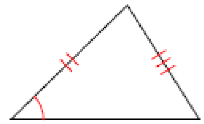
27. State whether the triangles are congruent and why (by which 'law'). Here, similar tick marks on sides or angles indicate that the sides or angles are congruent.

a.

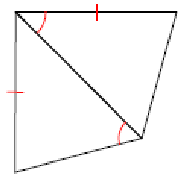


b.

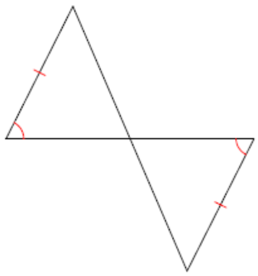




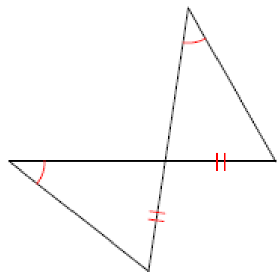
c.



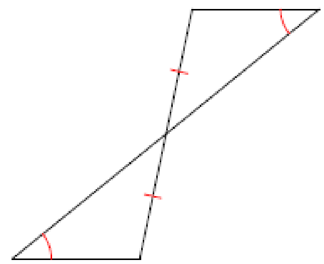
d.



e.

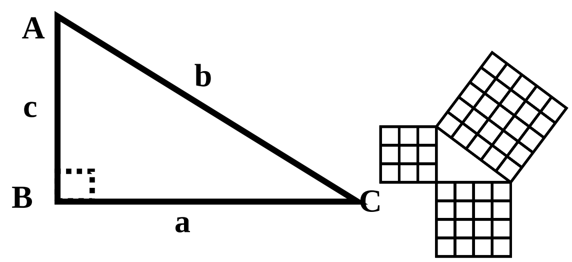


f.



28. **TRIGONOMETRY Introduction**. Recall from Grade 9 and 10 that Trigonometry is just the study of the measurements *(..metry*) of three (*tri…*) cornered (*..gon..*) figures. *ie*: The study of triangles. Once you understand triangles you understand every possible shape.

29. **Review of Pythagorean Theorem**.



" *the square on the longest side ('hypotenuse') of a right-angle triangle equals the sum of the squares on the other two sides*"
[Pythagoras of Samos: 580BC – 500BC]

in the diagram at right then

$\mathbf{b}^2 = \mathbf{a}^2 + \mathbf{c}^2$; where **b** is the hypotenuse by itself.

Recall that the hypotenuse is always across from the 90° corner.

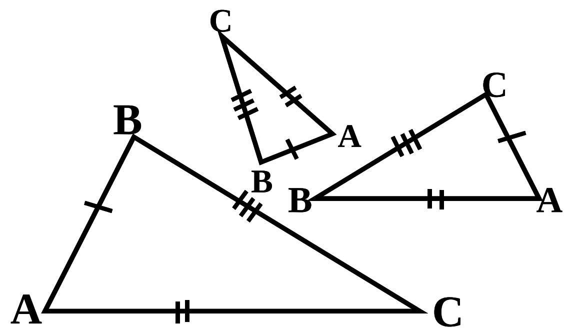
Recall how we label parts of triangles. Sides are little letters across from their corresponding capital letter corner. So, angle **A** has side '**a**' across from it, side '**b**' has angle **B** across from it, etc.

REVIEW OF SIMILAR TRIANGLES

30. Consult your notes or resources from prior grades to refresh your concept of similar triangles. If you understood similar triangles, then trigonometry is so much easier to understand. A very quick review follows.

Similar Triangles (*Review from Grade 9 and 10*)

31. Triangles are called '*similar*' if they have the same shape. The three triangles in the diagram to the right are similar triangles. They have the same shape (the same corner angles) and corresponding sides are *proportional* in length.



'Corresponding' sides of similar triangles are marked with similar number of 'tick' marks

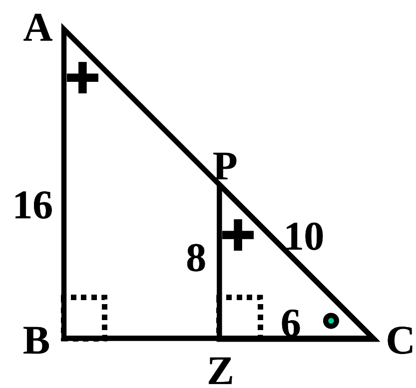
32. Further, if you know at least three parts of a triangle including at least one side then you know the shape **and the size** of that triangle.

Similar Triangles Calculation

33. Triangles **ABC** and **PZC** are '*similar'*. They have the same corner angles at **C**. They both have a **90º corner** and therefore the left over angles at **A** and **P** are the same. The three angles the same makes them similar. Consequently, their *corresponding sides* are *proportional*.

$\frac{AB}{PZ} = 2; \quad \frac{AC}{CP} = 2; \quad \frac{BC}{CZ} = 2$

So, side **AC** must be **20** and side **BC** must be **12**.

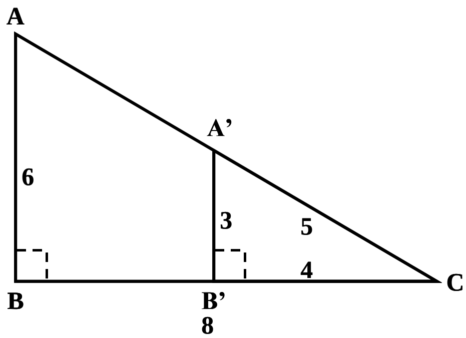


$\frac{16}{8} = \frac{20}{10} = \frac{12}{6}$ = **2.** All parts of the '**mommy**' triangle are just twice the '**baby**' triangle parts.

Triangles ABC and **PZC** are '*similar'* can be written as '**ΔABC~ΔPZC'** in basic notation.

34. **You try a few:**

a. Find all the missing line segments in this figure.

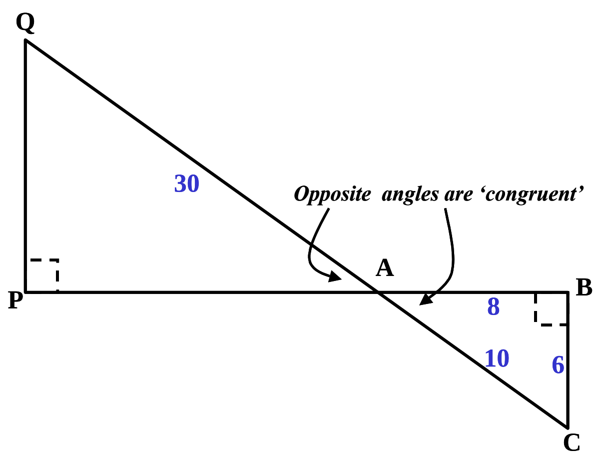


This could be rafters in a house or girders in a bridge!

Find: AC: ____________
Find BC: ____________
Find: AA’:____________
Find BB’:_______________

Our Solution:
(notice that we sometimes use B and B’ to mark ***corresponding corners****.* ***B’*** *is pronounced “****B prime****”)*

b. Find Length **PQ** and Length **AP** of these similar triangles.



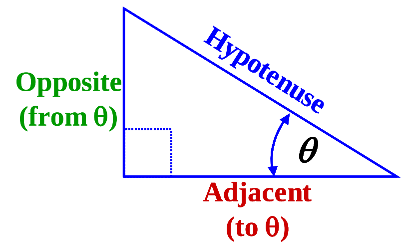
ΔABC and ΔAPQ are similar (ΔABC~ΔAPQ) since they have all three of the same angles.

Our Solution:

REVIEW OF THE TRIGONOMETRIC RATIOS OF RIGHT-ANGLE TRIANGLES

40. **Naming parts of a triangle**. We give the six measures of the triangles names using letters as above. We can also name them relative to a given acute angle corner as below.

41. Earthlings named certain comparisons of the sides of a triangle as the following '*ratios*':



a. tangent of angle θ = $\dfrac{Opposite}{Adjacent}$

b. **sine** of angle θ = $\dfrac{Opposite}{Hypotenuse}$

c. cosine of angle θ = $\dfrac{Adjacent}{Hypotenuse}$

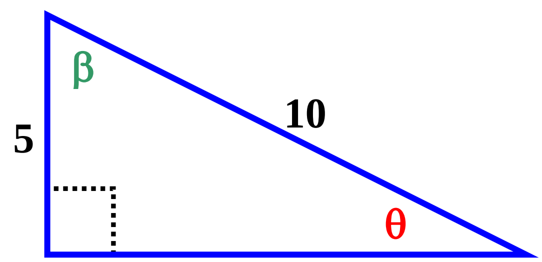
SO/H

CA/H

TO/A

MEMORIZE THESE ↑ RATIOS!!!!

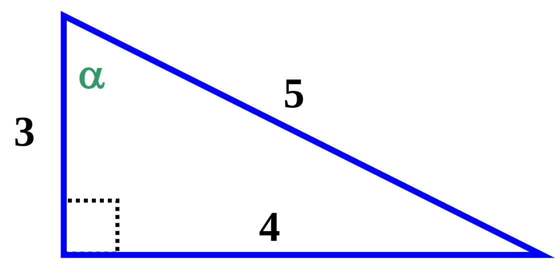
42. **Example 1**. Find the following trigonometric ratios for the triangle. Express as a decimal also.



a. $\boldsymbol{sin}\,\theta =$

b. $\boldsymbol{cos}\,\beta =$

43. **Example 2**. Find the following trigonometric ratios for the triangle. Express as a decimal also.



a. $sin\,\alpha =$

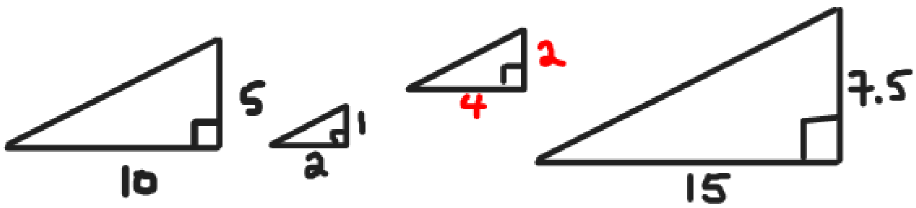
b. $cos(\alpha) =$

c. $tan\,\alpha =$

Knowing any trigonometric ratio of a right triangle tells you exactly its shape! You can readily draw any right-angle triangle by knowing any of its trigonometric ratios.

Example: Draw a triangle that has a tangent of 0.5.

$$\tan\angle = \frac{OPP}{ADJ} = 0.5 = 5/10 = 1/2 = 2/4 = 7.5/15$$



Lots of possibilities, but they all look like exactly the same triangle, just different sizes

GREEK ALPHABET

44. Notice that often mathematicians often prefer to use letters of the **Greek alphabet** to represent the measure of an **angle**. The most frequently used letters for angles are: 'θ' Theta; 'α' Alpha; and 'β' Beta.

45. **Using a calculator to find a trig ratio from a given angle**. Find the trigonometric ratio (to four decimal places) for the given angles.

a. **tan(30°)**

0.5773

b. **sin(30°)**

0.5000

c. **cos(30°)**

0.8660

d. **tan(47°)**

1.0724

e. **cos(72°)**

0.3090

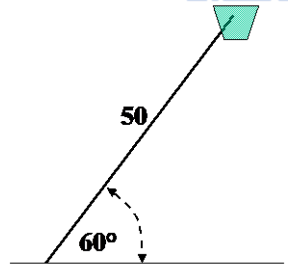
f. **sin(63°)**

0.8910

******Make sure your calculator shows it is in '*degree mode*' so that it uses angles measured in degrees. There is generally a **DRG** button that selects what units are used for angle calculations on your calculator.

****** You may also like to use the tables of trigonometric ratios that I have prepared for you at the back of these notes as an appendix if you have no calculator. We used tables calculated by our ancestors for a couple thousand years before calculators were invented!

46. **Solving triangle problem 1**. Jason is flying his kite, the string is 50 meters long and the string makes an angle of 60° (measured with his phone) with the flat ground. How high up is the kite above the ground? (do not include that Jason is 1.5 meters tall)



$$\sin(60) = \frac{height}{50}$$

$$\therefore h = 50\sin(60) = 50*0.866 = 43.3$$

Ans: **43.3** meters up

47. **Inverse trigonometric functions**. The trig functions give a trigonometric *value for* a related given *angle*. The '*inverse trig*' functions work backwards of that: they provide an *angle for* a related given trigonometric *value*.

Trig function →

°	sin	cos	tan
30	0.5000	0.8660	0.5774
35	0.5736	0.8192	0.7002
45	0.7071	0.7071	1.0000
50	0.7660	0.6428	1.1918

← Inverse Trig function

See the 'trig tables' at the end of these notes.

48. **On your calculator**. Use your calculator to find the following:

a. $\cos^{-1}(.707)$ =

b. $\sin^{-1}(.500)$ =

c. **arctan(1.1918)** =

d. **cos(45°) =**

e. **sin(30°) =**

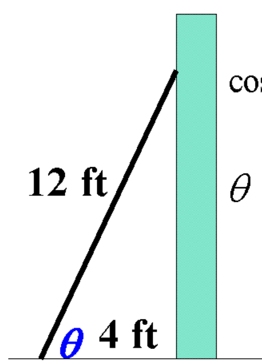
f. **tan(50°) =**

g. **arcsin(0.5) =**

h. **tan(45°)**

i. **arccos(2.5)** =

49. **Solving Triangle Problem 2**. A ladder leans against a wall. The ladder is 12 feet long and its feet are 4 feet from the wall. At what angle, θ, measured from the foot of the ladder is it leaning against the wall?



$$\cos(\theta) = \frac{Adj}{Hyp} = \frac{4}{12} = \frac{1}{3} = 0.333$$

$$\theta = \cos^{-1}(0.333) = 70.5^\circ$$

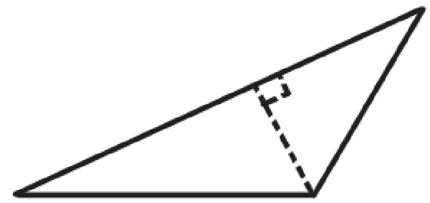
Ans: 70.5°

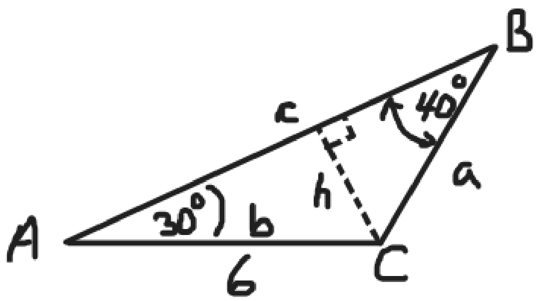
SOLVING TRIANGLES THAT ARE NOT RIGHT-ANGLE TRIANGLES

There are six parts to a triangle: three sides and three corner angles. **If you know any three parts of a triangle you can figure out the rest**! Even if it is **not** a right-angle triangle! (except for one case, can you figure out my little fib!?)

SOLVING TWO RIGHT TRIANGLES

Any triangle, obtuse, right, or acute can be broken up so that it is two right triangles!





Solve for side **a** of this obtuse triangle.

Find h using the left Δ:

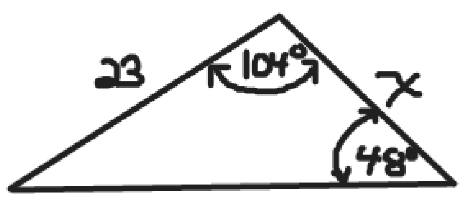
$h = 6 \sin 30° = 3$

Now find 'a'

$\sin 40° = \frac{3}{a}$

$\therefore \quad a = \frac{3}{\sin 40} \cong 4.67$

You try



Break it into two right triangles by *producing* a perpendicular line.

Cosine and Sine Laws

The Cosine Law and the Sine Law save you from having to solve two different triangles. With one simple relationship (formula) you can solve the main triangle without breaking it into two right triangles.

COSINE LAW

50. **The Cosine Law**. The cosine law can sometimes be used to solve a triangle given three other measures of a triangle. The '*cosine law*' is expressed mathematically as:

$a^2 = b^2 + c^2 - 2bc\,cos(A)$ where it is assumed that side little **a**, is opposite from angle big **A**.

51. In words the cosine law says that the square of an unknown side is equal to the sum of the squares of the other two sides less twice the product of the other two sides multiplied with the cosine of the angle between those two sides. Easy! lol.

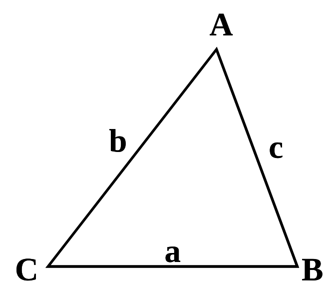
52. ~~The proof of this law will be left as an after-class exercise for those interested.~~

53. **Use of the Cosine Law**. The cosine law is used under two conditions:

a. given two sides of a triangle and the '*included angle*' between them to find the side opposite from the given included angle; or

b. given all three sides of a triangle to find any and all corner angles.

54. **Example**: Find side c given: a = 4; b = 4; and the angle between them (∠C) = 60°



$c^2 = a^2 + b^2 - 2*a*b*cos(\angle C)$

Evaluate (ie: plug in)

$c^2 = (4)^2 + (4)^2 - 2\,(4)(4)cos60°$

$c^2 = 16 + 16 - 32*0.5$

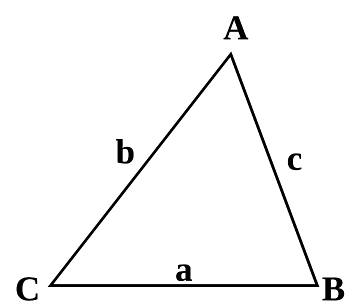
$c^2 = 32 - 16$

$c^2 = 16$

so, take the square root of both sides:

c = 4

55. **You Try**. Find side **c** given: a = 5; b = 7; and the angle between them (∠C) = 45°



$c^2 = a^2 + b^2 - 2*a*b*cos(\angle C)$

56. You might notice that the diagrams of the triangles above are not drawn accurately or to scale. If you are given values for the sides use them even though the diagram does not look like those are the lengths of the sides.

57. Changing the formula for finding different sides. The formula works regardless of what names you give your parts of the triangle. Provided you label the sides and corners in the normal manner then all three of these formulas are valid:

a. $\mathbf{a^2 = b^2 + c^2 - 2bc*cos(A)}$

b. $\mathbf{b^2 = a^2 + c^2 - 2ac*cos(B)}$

c. $\mathbf{c^2 = a^2 + b^2 - 2ab*cos(C)}$

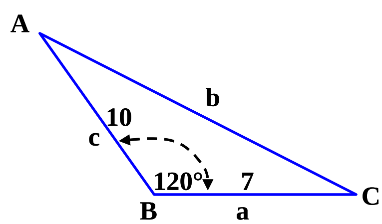
When you 'compose your formula' just make sure that:

- The side you want is by itself on the left of the '=' in the formula
- The other two sides follow the pattern of the formula on the right of the '='.
- The angle *across from the side* you want is in the cosine at the end of the formula. So, if you are looking for side little 'b' then big 'B' will be the angle at the end.

58. **Solving sides of obtuse triangles**. The cosine law works for obtuse triangles also. Obtuse triangles have an angle that is more than 90°.

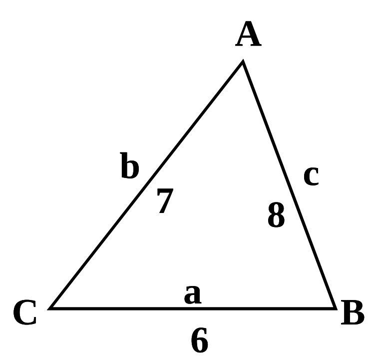
59. **Example for Obtuse Triangle**.

$\mathbf{b^2 = a^2 + c^2 - 2*a*c*cos(B)}$
$b^2 = (7)^2 + (10)^2 - 2(7)(10)(-0.5)$
$b^2 = 49 + 100 - (-70)$
Subtracting a negative is the same as adding
$b^2 = 149 + 70 = 219$
$b = \sqrt{219} \approx 14.8$



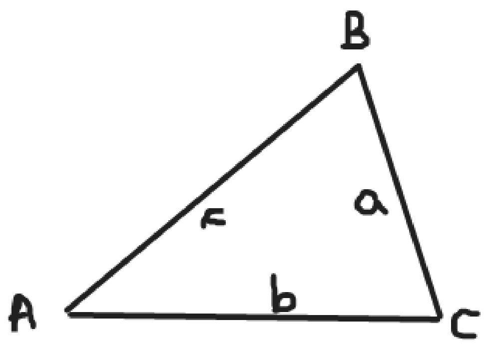
Notice when you take the cosine of an angle greater than 90° you can get a negative number! i.e.: a *negative negative* 70 is a positive 70. Of course, your calculator knows that if you are just smashing the entire expression in in one shot.

60. **Finding any angle given three sides of a triangle**. The cosine formulae can be used to find all three angles of a triangle given all three sides.
Example: (triangle at right)
Find angle C
$\mathbf{c^2 = a^2 + b^2 - 2*a*b*cos(C)}$
$8^2 = 6^2 + 7^2 - 2(6)(7)\cos(C)$
$64 = 36 + 49 - 84*\cos(C)$
$64 = 85 - 84*\cos(C)$
$-21 = -84*\cos(C)$
$\frac{-21}{-84} = cos(C) \quad \therefore \angle C = cos^{-1}(0.25) =$
75°



61. You notice the formula works quite nicely; it just takes a bit of algebra to solve for the angle. **You try** finding angle A and B above.

If you do not like doing algebra and manipulating symbols logically to produce other truths and facts and laws and valid conclusions, then just use these formulae below to find the corners of any triangle given three known sides. If you trust me that is! I prove these to you in a movie! Ask!



$$\angle A = \cos^{-1}\left(\frac{b^2 + c^2 - a^2}{2 \cdot b \cdot c}\right)$$

$$\angle B = \cos^{-1}\left(\frac{a^2 + c^2 - b^2}{2 \cdot a \cdot c}\right)$$

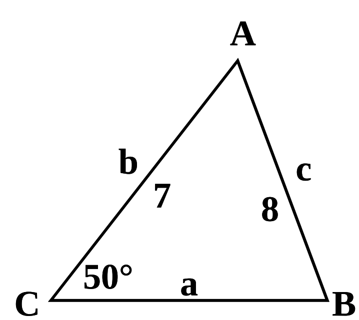
$$\angle C = \cos^{-1}\left(\frac{a^2 + b^2 - c^2}{2 \cdot a \cdot c}\right)$$

THE SINE LAW

62. The sine law is the other law that works on triangles that are not right-angle. If the cosine law does not work, the sine law will. Further with these two laws: cosine law and sine law, once you are given **any three parts** of a triangle you can **figure out the other three parts**. (except for one case).

63. The sine law is actually three separate and equal proportions which we show like this: $\frac{\boldsymbol{a}}{\boldsymbol{\sin A}} = \frac{\boldsymbol{b}}{\boldsymbol{\sin B}} = \frac{\boldsymbol{c}}{\boldsymbol{\sin C}}$; the proof of this law will be given if time permits.

64. **Example of Sine Law**. Given the triangle at right find the measure of angle B.



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{a?}{\sin A\,?} = \frac{7}{\sin B\,?} = \frac{8}{\sin 50°}$$

Select the pair that will solve for B

$\frac{7}{\sin B} = \frac{8}{\sin 50°}$ so, rearranging by cross multiplying or algebra:

$$\sin B = \frac{7 * \sin 50°}{8} = 0.6703$$

$$\angle B = \sin^{-1}(0.6703) = 42.1°$$

65. Now you find side ‘a’ above if you are given that ∠A=87.9°.

Of course, there are lots of readily available apps and websites to do all these calculations for you!!

IMPOSSIBLE TRIANGLES

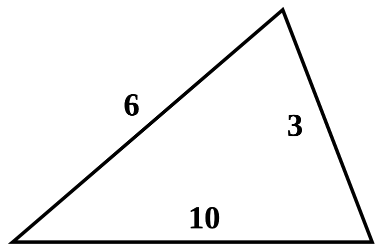
66. Be careful with triangle problems, especially those involving the Sine Law. It is readily possible to contrive impossible triangles when given two angles and a side or even when given all three sides.

‘TRIANGLE INEQUALITY’ EXAMPLE

67. Can you see that this triangle is impossible for the given side lengths?

Have you ever tried to make a triangle where one side was longer than the sum of the other two sides?

Of course, if we always drew our triangles nicely to scale, we would readily notice this, but we seldom actually sketch triangles to proper relative scale.



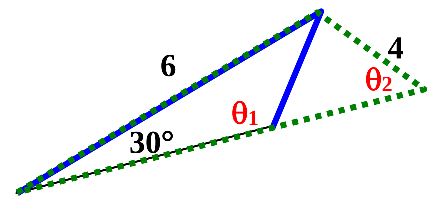
Make sure that the sum of any two sides is longer than the remaining side!

AMBIGUITY CASE WITH SINE LAW

69. There is a special case of ambiguity (meaning ‘*uncertain*’) when ***two sides and a non-included angle*** are given. What can happen is that there are two answers! So, you are uncertain which answer is correct; but in fact both of them are correct. **Technically you are not required to know the ambiguous case of the sine law** in Essential Math, but you will encounter it in related support resources so it is presented here regardless.

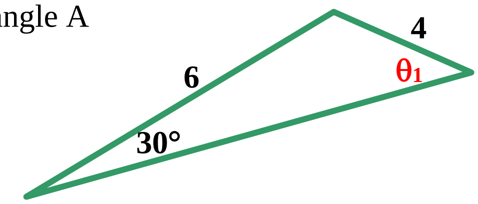
70. Given a triangle that has a side of 4 and an angle opposite from that side of 30° and another side of length 6, find the angle, **θ,** opposite the 6.

71. Sketch the problem! You will discover there are two possible triangles that match that description!!



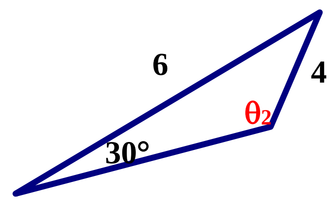
Triangle A

72. A calculator will tell you Triangle A has a corner angle **θ** such that **θ = sin^{-1}(3/4)**
So that **θ = 48.6°**



Triangle B

73. But did you know there is another angle that has a sine of ¾. That angle is (180 – 48.6) or **131.4°**



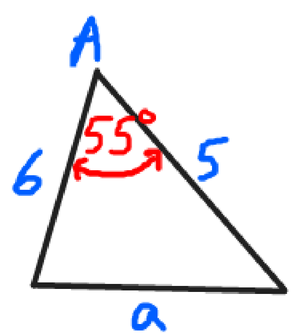
CONDITIONS FOR AMBIGUOUS CASE SOLUTIONS

74. Ambiguous means: ‘having more than one meaning’. The ambiguous case where there are two possible triangles that will satisfy a certain description will happen **only if** you are given two sides and a non-included angle. Just think of sticking your fishing rod in between two rocks at a certain angle and your rod and fishing line being a certain length. The fishing line may be toward you or away from you making a different angle at the surface of the river and the tip of the pole. The possible angle solutions are supplementary however.

Some more worked examples

EXAMPLE 1

75. Label the triangle with letters.
Select the cosine law since the problem is two sides given with an included angle between them.
In this case with the letters used here:



$a^2 = b^2 + c^2 – 2bc*\cos(A)$

$a^2 = (6)^2 + (5)^2 –2(6)(5)*\cos(55°)$

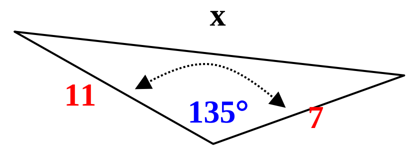
$a^2 = 12 + 10 – 30* 0.574$

$a^2 = 22 – 17.2 = 4.8$

$c = \sqrt{4.8} \cong 2.2$

Hint: When evaluating; always substitute in using brackets to preclude confusion.

EXAMPLE 2 COSINE LAW – OBTUSE TRIANGLE



76. No matter how you label your triangle and write down the formula you will get:

$x^2 = (7)^2 + (11)^2 - 2(7)(11)\cos(135°)$

$x^2 = 49 + 121 - 154*(-\mathbf{0.707})$

$x^2 = 170 + 108.9 = 278.9$

$$x = \sqrt{278.9} \cong 16.7$$

Note: Don't forget subtracting the negative is the same as adding. The cosine of an obtuse angle is negative.

Hint: if you came up with an answer of 22 for x would that make sense?

EXAMPLE 3 – FINDING AN ANGLE USING COSINE LAW

77. Possible when given all three sides of a triangle.

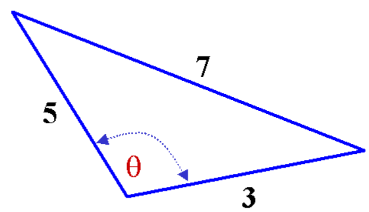
Label your triangle with letters

Depending on how you label your triangle your formula will eventually read:

$\mathbf{7^2 = (3)^2 + (5)^2 - 2(3)(5)*\cos(\theta)}$

Notice we selected the formula so that the side opposite the unknown angle is alone on the left of the '='.

Simplifying and solving with algebra



$49 = 9 + 25 - 30*\cos(\theta)$

$49 = 34 - 30*\cos(\theta)$

$15 = -30 * \cos(\theta)$

$-0.5 = \cos(\theta)$ (Notice the **negative** sign)

$\theta = \cos^{-1}(-0.5) = \mathbf{120°}$

The Sine Law

If the Cosine Law does not work, then the Sine Law will!

The Sine Law requires that you know a **side & angle pair**. So, if you have a **known side and a known angle across from that side** you are good to go if you have one other element of the triangle.

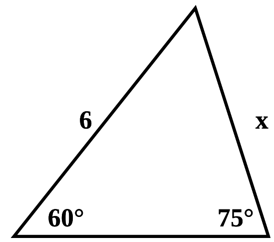
Sine Law: Finding a Side

78. Label your triangle with letters. Set up the formulae:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Regardless how you label them select a pair so that you have the unknown and three givens and you will get:

$$\frac{6}{\sin 75°} = \frac{x}{\sin 60°}$$



so $x = \frac{6 * \sin 60°}{\sin 75°} \cong 5.38$

Notice: Widest angles should have longest opposite sides! Sharpest angles should have shortest opposite sides!

Sine Law: Finding an Angle.

79. Label your triangle with letters. Set up the formulae:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

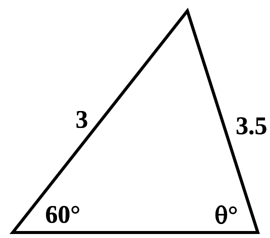
Regardless of how you label them select a pair that you can have an unknown and you will get:

$\frac{\mathbf{3}}{\boldsymbol{\sin\theta}} = \frac{\mathbf{3.5}}{\boldsymbol{\sin 60°}}$ so, isolate the sin θ using algebra (cross multiply).

$$\frac{3}{\sin\theta} = \frac{3.5}{\sin 60°}$$

$$\sin\theta = \frac{3 \cdot \sin 60°}{3.5} = 0.7423$$

$$\theta = \sin^{-1}(0.7423) = 47.93° \doteq 48°$$



so: $\sin\theta = \frac{3\sin 60}{3.5} = \frac{3*0.866}{3.5} = 0.742$

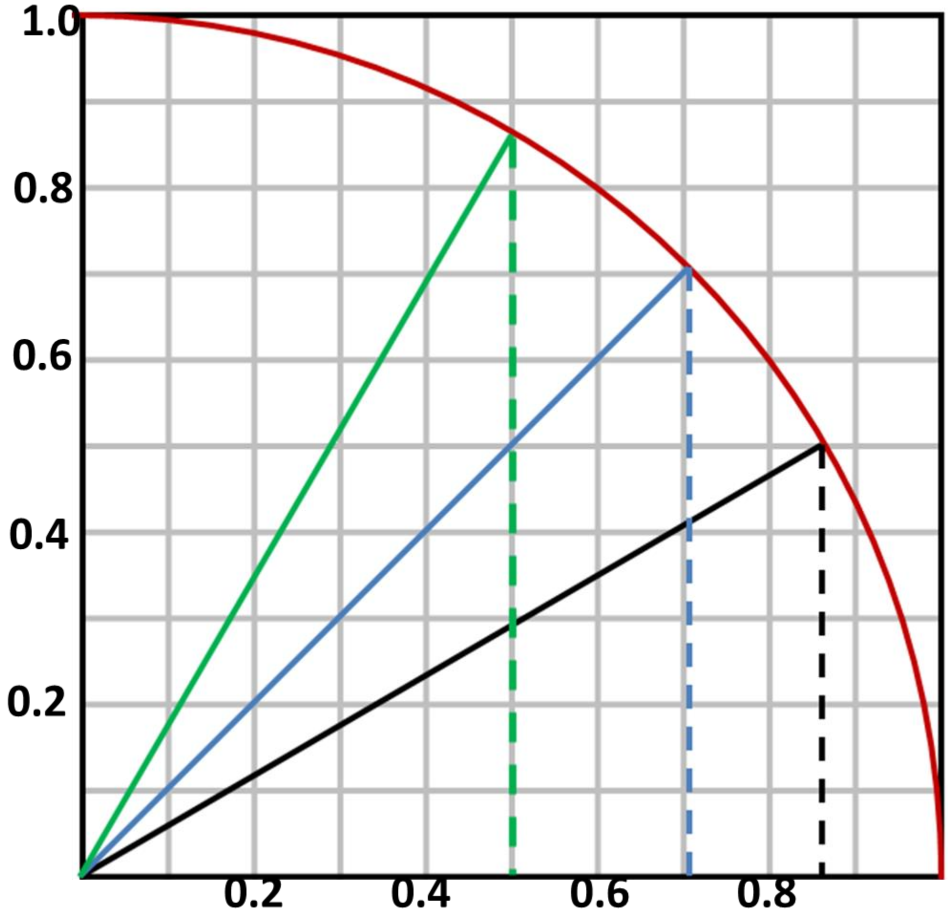
so θ = $\sin^{-1}(0.742)$ = **47.93° = 48°**

we often just round angles to the nearest degree since a degree is such a tiny sliver.

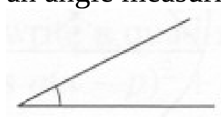
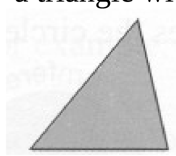
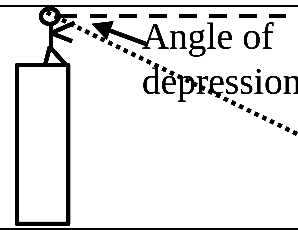
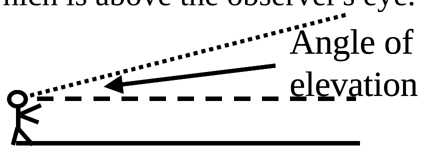
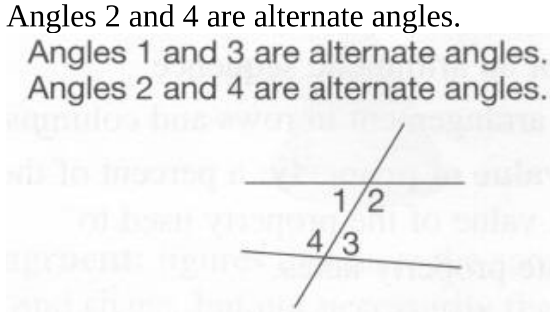
Don't do this in astronomy though with enormous distances!

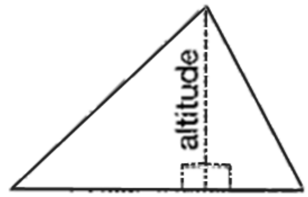
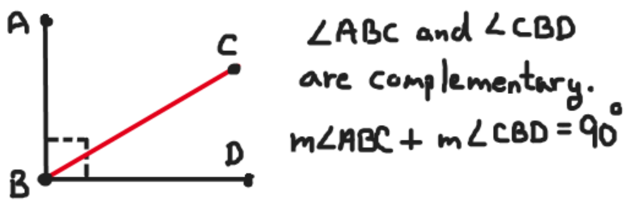
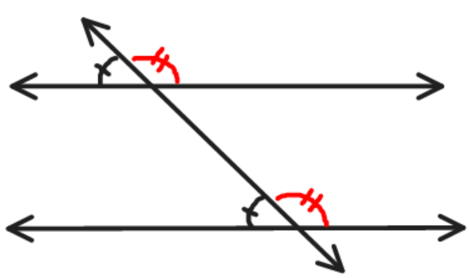
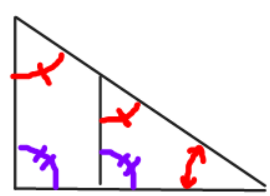
TRIGONOMETRIC RATIOS AS CENTRAL ANGLES ON THE ARC OF A CIRCLE

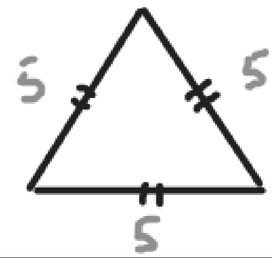
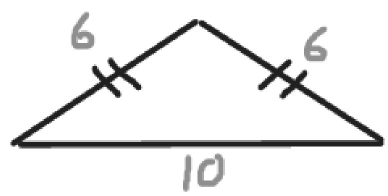
80. If you did not like studying triangles, maybe this sector of a circle will help! Get your teacher to explain this. This is actually much easier to understand and to estimate the trigonometric ratios. It turns out that the **sine** of an angle is just how **high** up you are on the circle, the **cosine** is just how far to the **right** you are on the circle, and the tangent is just how high up you are on the very right edge of the diagram! **omg**.

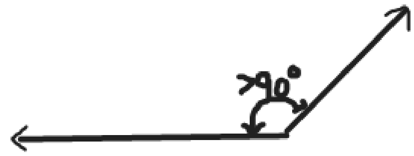
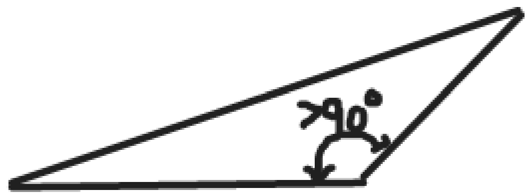
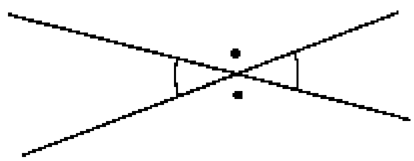
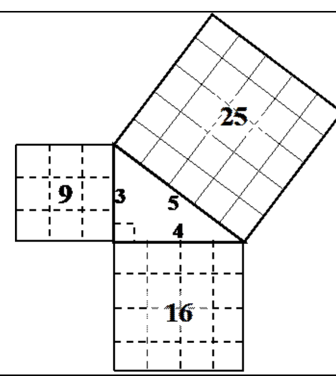


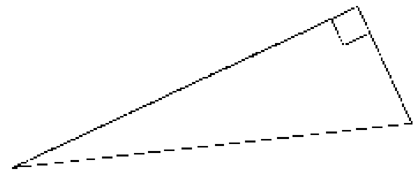
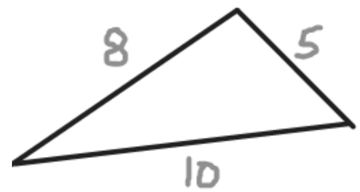
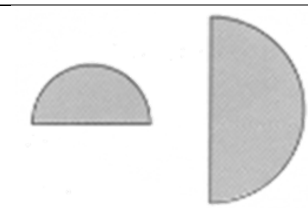
APPENDIX A – TRIGONOMETRY GLOSSARY (GRADE 10 NOTES)

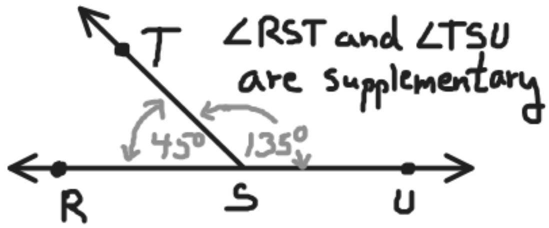
acute angle	an angle measuring less than 90° 
acute triangle	a triangle with all three acute angles 
Angle of depression:	the angle formed between the horizon line and the line of sight to the object, which is below the observer's eye. 
Angle of elevation	the angle formed between the horizontal plane and the line of sight to the object, which is above the observer's eye.  *Notice if you are looking down at someone and they are looking up at you that your angle of depression to them is their **angle of elevation** to you!*
alternate angles	angles that are between two lines and are on opposite sides of a transversal that cuts the two lines Angles 1 and 3 are alternate angles Angles 2 and 4 are alternate angles. 

altitude	the **perpendicular** distance from the base of a figure to the opposite side or vertex; also the height of an aircraft above the ground  altitude is always measured **perpendicular** to some base line
complementary angles	two angles whose sum is 90º ∠ABC and ∠CBD are complementary angles.  Not compl*i*mentary, as in: "don't you look nice today ∠ABC!"
congruent	figures that have the same size and shape, but not necessarily the same orientation 
corresponding angles in similar triangles and transversals cutting parallel lines	two angles that are equal (congruent) and 'match up' to each other.  
cosine	for an acute ∠*A* in a right triangle, the ratio of the length of the side adjacent to ∠*A*, to the length of the hypotenuse $cos(\angle A) = \frac{length of side adjacent to \angle A}{length of hypotenuse side} = \frac{adj}{hyp}$

domain of a function	the set of x-values (or valid input numbers) represented by the graph or the equation of a function
equilateral triangle	a triangle with three equal sides 
function	An expression that performs an operation on a given number: Eg: the function 'f(x)' or 'f of x' or 'f at x' given by: ***f(x)= 2x +5*** just takes any given number 'x' doubles it and then adds five.
Inverse trig function	The inverse trigonometric functions of $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$ 'undo' their respective functions. They tell you the angle that goes with any particular trigonometric ratio. **Example**: if you want to know what the angle θ for a corner that has a sine of 0.5 you use: The angle θ is = $\sin^{-1}(0.5)$; and it tells you the angle is 30 degrees. Also called arc sin, arccos, and arctan.
irrational number	a number that **cannot** be written in the form m/n where *m* and *n* are integers *(n* $\neq$ 0). Eg: $\sqrt{2}, \sqrt{5}$, π, ….
isosceles acute triangle	a triangle with at least two equal sides and all angles less than 90°
isosceles obtuse triangle	a triangle with at least two equal sides and one angle greater than 90°
isosceles right triangle	a triangle with two equal sides and a 90° angle
isosceles triangle	a triangle with at least two equal sides [Latin: iso → same, scele → side] 

legs of a right-angle triangle	the sides of a right triangle that form the right angle
obtuse angle	an angle greater than 90° and less than 180° 
obtuse triangle	a triangle with one angle greater than 90°. 
opposite angles	the equal angles that are formed by two intersecting lines. (Also called **vertical** angles for some reason) 
plane geometry	the study of two-dimensional figures; that is, figures drawn or visualized on a plane
proportion	a statement that two ratios are equal
Pythagorean Theorem	for any right triangle, the area of the square on the hypotenuse is equal to the sum of the areas of the squares on the other two sides or symbolically: $c^2 = a^2 + b^2$; **provided c** is the hypotenuse. 
radical	the root of a number; for example, $-\sqrt{400}$, $\sqrt[3]{5}$, and so on
range of a function	the set of y-values (or output numbers) represented by the graph or the equation of a function
right angle	a 90° angle.

right triangle	a triangle that has a right (square) angle. 
scalene triangle	a triangle with no two sides equal 
similar figures	figures with the same shape, but not necessarily the same size 
sine	for an acute $\angle A$ in a right triangle, the ratio of the length of the side opposite $\angle A$, to the length of the hypotenuse $\sin(\angle A) = \frac{\textit{length of side opposite to } \angle A}{\textit{length of hypotenuse side}} = \frac{opp}{hyp}$
square of a number	the product of a number multiplied by itself; for example, 25 is the square of 5.
square root	a number which, when multiplied by itself, results in a given number; for example, 5 and -5 are the square roots of 25, since 5 * 5 is 25 (and (-5)*(-5) is also 25). A ‘square root’ undoes a ‘square’!
Sum of angles in a triangle	One of the important properties of a triangle. The sum of the angles in a triangle is 180°. Proved in Grade 9.

supplementary angles	two angles whose sum is 180°. Supplementary angles for a **linear pair**. 
tangent	for an acute ∠*A* in a right triangle, the ratio of the length of the side opposite ∠*A*, to the length of the side adjacent to ∠*A* $\tan(\angle A) = \frac{length\, of\, side\, opposite\, to \angle A}{length\, of\, side\, adjacent\, to \angle A} = \frac{opp}{adj}$

Add in your own definitions here if you want:

APPENDIX B: TRIGONOMETRIC VALUES FOR ANGLES 0° TO 90°

Angle	Sin	Cos	Tan	Angle	Sin	Cos	Tan
0	0	1	0	46	0.7193	0.6947	1.036
1	0.0175	0.9998	0.0175	47	0.7314	0.6820	1.072
2	0.0349	0.9994	0.0349	48	0.7431	0.6691	1.111
3	0.0523	0.9986	0.0524	49	0.7547	0.6561	1.150
4	0.0698	0.9976	0.0699	50	0.7660	0.6428	1.192
5	0.0872	0.9962	0.0875	51	0.7771	0.6293	1.235
6	0.1045	0.9945	0.1051	52	0.7880	0.6157	1.280
7	0.1219	0.9925	0.1228	53	0.7986	0.6018	1.327
8	0.1392	0.9903	0.1405	54	0.8090	0.5878	1.376
9	0.1564	0.9877	0.1584	55	0.8192	0.5736	1.428
10	0.1736	0.9848	0.1763	56	0.8290	0.5592	1.483
11	0.1908	0.9816	0.1944	57	0.8387	0.5446	1.540
12	0.2079	0.9781	0.2126	58	0.8480	0.5299	1.600
13	0.2250	0.9744	0.2309	59	0.8572	0.5150	1.664
14	0.2419	0.9703	0.2493	60	0.8660	0.5000	1.732
15	0.2588	0.9659	0.2679	61	0.8746	0.4848	1.804
16	0.2756	0.9613	0.2867	62	0.8829	0.4695	1.881
17	0.2924	0.9563	0.3057	63	0.8910	0.4540	1.963
18	0.3090	0.9511	0.3249	64	0.8988	0.4384	2.050
19	0.3256	0.9455	0.3443	65	0.9063	0.4226	2.145
20	0.3420	0.9397	0.3640	66	0.9135	0.4067	2.246
21	0.3584	0.9336	0.3839	67	0.9205	0.3907	2.356
22	0.3746	0.9272	0.4040	68	0.9272	0.3746	2.475
23	0.3907	0.9205	0.4245	69	0.9336	0.3584	2.605
24	0.4067	0.9135	0.4452	70	0.9397	0.3420	2.747
25	0.4226	0.9063	0.4663	71	0.9455	0.3256	2.904
26	0.4384	0.8988	0.4877	72	0.9511	0.3090	3.078
27	0.4540	0.8910	0.5095	73	0.9563	0.2924	3.271
28	0.4695	0.8829	0.5317	74	0.9613	0.2756	3.487
29	0.4848	0.8746	0.5543	75	0.9659	0.2588	3.732
30	0.5000	0.8660	0.5774	76	0.9703	0.2419	4.011
31	0.5150	0.8572	0.6009	77	0.9744	0.2250	4.331
32	0.5299	0.8480	0.6249	78	0.9781	0.2079	4.705
33	0.5446	0.8387	0.6494	79	0.9816	0.1908	5.145
34	0.5592	0.8290	0.6745	80	0.9848	0.1736	5.671
35	0.5736	0.8192	0.7002	81	0.9877	0.1564	6.314
36	0.5878	0.8090	0.7265	82	0.9903	0.1392	7.115
37	0.6018	0.7986	0.7536	83	0.9925	0.1219	8.144
38	0.6157	0.7880	0.7813	84	0.9945	0.1045	9.514
39	0.6293	0.7771	0.8098	85	0.9962	0.0872	11.43
40	0.6428	0.7660	0.8391	86	0.9976	0.0698	14.30
41	0.6561	0.7547	0.8693	87	0.9986	0.0523	19.08
42	0.6691	0.7431	0.9004	88	0.9994	0.0349	28.64
43	0.6820	0.7314	0.9325	89	0.9998	0.0175	57.290
44	0.6947	0.7193	0.9657	89.5	1.0000	0.0087	114.589
45	0.7071	0.7071	1.0000	90	1	0	Infinity