Chapter 2

Techniques of Counting

INTRODUCTION

In this chapter we develop some techniques for determining without direct enumeration the number of possible outcomes of a particular experiment or the number of elements in a particular set. Such techniques are sometimes referred to as combinatorial analysis.

FUNDAMENTAL PRINCIPLE OF COUNTING

We begin with the following basic principle.

Fundamental Principle of Counting: If some procedure can be performed in n_1 different ways, and if, following this procedure, a second procedure can be performed in n_2 different ways, and if, following this second procedure, a third procedure can be performed in n_3 different ways, and so forth; then the number of ways the procedures can be performed in the order indicated is the product $n_1 \cdot n_2 \cdot n_3 \dots$.

Example 2.1: Suppose a license plate contains two distinct letters followed by three digits with the first digit not zero. How many different license plates can be printed?

The first letter can be printed in 26 different ways, the second letter in 25 different ways (since the letter printed first cannot be chosen for the second letter), the first digit in 9 ways and each of the other two digits in 10 ways. Hence

$$26 \cdot 25 \cdot 9 \cdot 10 \cdot 10 = 585{,}000$$

different plates can be printed.

FACTORIAL NOTATION

The product of the positive integers from 1 to n inclusive occurs very often in mathematics and hence is denoted by the special symbol $n!$ (read "n factorial"):

$$n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-2)(n-1)n$$

It is also convenient to define $0! = 1$.

Example 2.2:

$$2! = 1 \cdot 2 = 2, \quad 3! = 1 \cdot 2 \cdot 3 = 6, \quad 4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24,$$
$$5! = 5 \cdot 4! = 5 \cdot 24 = 120, \quad 6! = 6 \cdot 5! = 6 \cdot 120 = 720$$

Example 2.3:

$$\frac{8!}{6!} = \frac{8 \cdot 7 \cdot 6!}{6!} = 8 \cdot 7 = 56 \qquad 12 \cdot 11 \cdot 10 = \frac{12 \cdot 11 \cdot 10 \cdot 9!}{9!} = \frac{12!}{9!}$$

PERMUTATIONS

An arrangement of a set of n objects in a given order is called a *permutation* of the objects (taken all at a time). An arrangement of any $r \leq n$ of these objects in a given order is called an *r-permutation* or a *permutation of the n objects taken r at a time.*

Example 2.4: Consider the set of letters a, b, c and d. Then:

(i) *bdca*, *dcba* and *acdb* are permutations of the 4 letters (taken all at a time);

(ii) *bad*, *adb*, *cbd* and *bca* are permutations of the 4 letters taken 3 at a time;

(iii) *ad*, *cb*, *da* and *bd* are permutations of the 4 letters taken 2 at a time.

The number of permutations of n objects taken r at a time will be denoted by

$$P(n, r)$$

Before we derive the general formula for $P(n,r)$ we consider a special case.

Example 2.5: Find the number of permutations of 6 objects, say a, b, c, d, e, f, taken three at a time. In other words, find the number of "three letter words" with distinct letters that can be formed from the above six letters.

Let the general three letter word be represented by three boxes:

Now the first letter can be chosen in 6 different ways; following this, the second letter can be chosen in 5 different ways; and, following this, the last letter can be chosen in 4 different ways. Write each number in its appropriate box as follows:

6	5	4

Thus by the fundamental principle of counting there are $6 \cdot 5 \cdot 4 = 120$ possible three letter words without repetitions from the six letters, or there are 120 permutations of 6 objects taken 3 at a time. That is,

$$P(6,3) = 120$$

The derivation of the formula for $P(n,r)$ follows the procedure in the preceding example. The first element in an r-permutation of n-objects can be chosen in n different ways; following this, the second element in the permutation can be chosen in $n-1$ ways; and, following this, the third element in the permutation can be chosen in $n-2$ ways. Continuing in this manner, we have that the rth (last) element in the r-permutation can be chosen in $n-(r-1) = n-r+1$ ways. Thus

Theorem 2.1: $P(n,r) = n(n-1)(n-2)\cdots(n-r+1) = \dfrac{n!}{(n-r)!}$

The second part of the formula follows from the fact that

$$n(n-1)(n-2)\cdots(n-r+1) = \frac{n(n-1)(n-2)\cdots(n-r+1)\cdot(n-r)!}{(n-r)!} = \frac{n!}{(n-r)!}$$

In the special case that $r = n$, we have

$$P(n,n) = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1 = n!$$

Namely,

Corollary 2.2: There are $n!$ permutations of n objects (taken all at a time).

Example 2.6: How many permutations are there of 3 objects, say, a, b and c?

By the above corollary there are $3! = 1\cdot 2\cdot 3 = 6$ such permutations. These are *abc, acb, bac, bca, cab, cba*.

PERMUTATIONS WITH REPETITIONS

Frequently we want to know the number of permutations of objects some of which are alike, as illustrated below. The general formula follows.

Theorem 2.3: The number of permutations of n objects of which n_1 are alike, n_2 are alike, $\ldots, n_r$ are alike is

$$\frac{n!}{n_1!\,n_2!\cdots n_r!}$$

We indicate the proof of the above theorem by a particular example. Suppose we want to form all possible 5 letter words using the letters from the word DADDY. Now there are $5! = 120$ permutations of the objects D_1, A, D_2, D_3, Y where the three D's are distinguished. Observe that the following six permutations

$$D_1D_2D_3AY, \quad D_2D_1D_3AY, \quad D_3D_1D_2AY, \quad D_1D_3D_2AY, \quad D_2D_3D_1AY, \quad D_3D_2D_1AY$$

produce the same word when the subscripts are removed. The 6 comes from the fact that there are $3! = 3 \cdot 2 \cdot 1 = 6$ different ways of placing the three D's in the first three positions in the permutation. This is true for each of the other possible positions in which the D's appear. Accordingly there are

$$\frac{5!}{3!} = \frac{120}{6} = 20$$

different 5 letter words that can be formed using the letters from the word DADDY.

Example 2.7: How many different signals, each consisting of 8 flags hung in a vertical line, can be formed from a set of 4 indistinguishable red flags, 3 indistinguishable white flags, and a blue flag? We seek the number of permutations of 8 objects of which 4 are alike (the red flags) and 3 are alike (the white flags). By the above theorem, there are

$$\frac{8!}{4!\,3!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 1} = 280$$

different signals.

ORDERED SAMPLES

Many problems in combinatorial analysis and, in particular, probability are concerned with choosing a ball from an urn containing n balls (or a card from a deck, or a person from a population). When we choose one ball after another from the urn, say r times, we call the choice an ordered sample of size r. We consider two cases:

(i) *Sampling with replacement.* Here the ball is replaced in the urn before the next ball is chosen. Now since there are n different ways to choose each ball, there are by the fundamental principle of counting

$$\overbrace{n \cdot n \cdot n \cdots n}^{r \text{ times}} = n^r$$

different ordered samples with replacement of size r.

(ii) *Sampling without replacement.* Here the ball is not replaced in the urn before the next ball is chosen. Thus there are no repetitions in the ordered sample. In other words, an ordered sample of size r without replacement is simply an r-permutation of the objects in the urn. Thus there are

$$P(n,r) = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

different ordered samples of size r without replacement from a population of n objects.

Example 2.8: In how many ways can one choose three cards in succession from a deck of 52 cards (i) with replacement, (ii) without replacement? If each card is replaced in the deck before the next card is chosen, then each card can be chosen in 52 different ways. Hence there are

$$52 \cdot 52 \cdot 52 = 52^3 = 140{,}608$$

different ordered samples of size 3 with replacement.

On the other hand if there is no replacement, then the first card can be chosen in 52 different ways, the second card in 51 different ways, and the third and last card in 50 different ways. Thus there are

$$52 \cdot 51 \cdot 50 = 132{,}600$$

different ordered samples of size 3 without replacement.

BINOMIAL COEFFICIENTS AND THEOREM

The symbol $\binom{n}{r}$, read "nCr", where r and n are positive integers with $r \leq n$, is defined as follows:

$$\binom{n}{r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{1\cdot 2\cdot 3\cdots(r-1)r}$$

These numbers are called the *binomial coefficients* in view of Theorem 2.5 below.

Example 2.9: $\binom{8}{2} = \frac{8\cdot 7}{1\cdot 2} = 28 \qquad \binom{9}{4} = \frac{9\cdot 8\cdot 7\cdot 6}{1\cdot 2\cdot 3\cdot 4} = 126 \qquad \binom{12}{5} = \frac{12\cdot 11\cdot 10\cdot 9\cdot 8}{1\cdot 2\cdot 3\cdot 4\cdot 5} = 792$

Observe that $\binom{n}{r}$ has exactly r factors in both the numerator and denominator. Also,

$$\binom{n}{r} = \frac{n(n-1)\cdots(n-r+1)}{1\cdot 2\cdot 3\cdots(r-1)r} = \frac{n(n-1)\cdots(n-r+1)(n-r)!}{1\cdot 2\cdot 3\cdots(r-1)r(n-r)!} = \frac{n!}{r!\,(n-r)!}$$

Using this formula and the fact that $n-(n-r) = r$, we obtain the following important relation.

Lemma 2.4: $\binom{n}{n-r} = \binom{n}{r}$ or, in other words, if $a+b=n$ then $\binom{n}{a} = \binom{n}{b}$.

Example 2.10: $\binom{10}{7} = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} = 120 \quad$ or $\quad \binom{10}{7} = \binom{10}{3} = \frac{10\cdot 9\cdot 8}{1\cdot 2\cdot 3} = 120$

Note that the second method saves both space and time.

Remark: Motivated by the second formula for $\binom{n}{r}$ and the fact that $0! = 1$, we define:

$$\binom{n}{0} = \frac{n!}{0!\,n!} = 1 \quad \text{and, in particular,} \quad \binom{0}{0} = \frac{0!}{0!\,0!} = 1$$

The Binomial Theorem, which is proved (Problem 2.18) by mathematical induction, gives the general expression for the expansion of $(a+b)^n$.

Theorem 2.5 (Binomial Theorem):

$$\begin{aligned}(a+b)^n &= \sum_{r=0}^{n} \binom{n}{r} a^{n-r}b^r \\ &= a^n + na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \cdots + nab^{n-1} + b^n\end{aligned}$$

Example 2.11:

$$\begin{aligned}(a+b)^5 &= a^5 + 5a^4b + \frac{5\cdot 4}{1\cdot 2}a^3b^2 + \frac{5\cdot 4}{1\cdot 2}a^2b^3 + 5ab^4 + b^5 \\ &= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5\end{aligned}$$

$$\begin{aligned}(a+b)^6 &= a^6 + 6a^5b + \frac{6\cdot 5}{1\cdot 2}a^4b^2 + \frac{6\cdot 5\cdot 4}{1\cdot 2\cdot 3}a^3b^3 + \frac{6\cdot 5}{1\cdot 2}a^2b^4 + 6ab^5 + b^6 \\ &= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6\end{aligned}$$

The following properties of the expansion of $(a+b)^n$ should be observed:

(i) There are $n+1$ terms.

(ii) The sum of the exponents of a and b in each term is n.

(iii) The exponents of a decrease term by term from n to 0; the exponents of b increase term by term from 0 to n.

(iv) The coefficient of any term is $\binom{n}{k}$ where k is the exponent of either a or b. (This follows from Lemma 2.4.)

(v) The coefficients of terms equidistant from the ends are equal.

We remark that the coefficients of the successive powers of $a+b$ can be arranged in a triangular array of numbers, called Pascal's triangle, as follows:

$$\begin{aligned}
(a+b)^0 &= 1 \\
(a+b)^1 &= a + b \\
(a+b)^2 &= a^2 + 2ab + b^2 \\
(a+b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\
(a+b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \\
(a+b)^5 &= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 \\
(a+b)^6 &= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6 \\
&\cdots\cdots\cdots
\end{aligned}$$

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad \textcircled{10} \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad \textcircled{15} \quad \textcircled{20} \quad 15 \quad 6 \quad 1 \\
\cdots\cdots\cdots
\end{array}$$

Pascal's triangle has the following interesting properties.

(*a*) The first number and the last number in each row is 1.

(*b*) Every other number in the array can be obtained by adding the two numbers appearing directly above it. For example: $10 = 4+6$, $15 = 5+10$, $20 = 10+10$.

We note that property (*b*) above is equivalent to the following theorem about binomial coefficients.

Theorem 2.6: $\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$

Now let $n_1, n_2, \ldots, n_r$ be nonnegative integers such that $n_1 + n_2 + \cdots + n_r = n$. Then the expression $\binom{n}{n_1, n_2, \ldots, n_r}$ is defined as follows:

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1!\, n_2! \cdots n_r!}$$

For example,

$$\binom{7}{2, 3, 2} = \frac{7!}{2!3!2!} = 210 \qquad \binom{8}{4, 2, 2, 0} = \frac{8!}{4!2!2!0!} = 420$$

These numbers are called the *multinomial coefficients* in view of the following theorem which generalizes the binomial theorem.

Theorem 2.7: $(a_1 + a_2 + \cdots + a_r)^n = \sum_{n_1 + n_2 + \cdots + n_r = n} \binom{n}{n_1, n_2, \ldots, n_r} a_1^{n_1} a_2^{n_2} \cdots a_r^{n_r}$

COMBINATIONS

Suppose we have a collection of n objects. A *combination* of these n objects *taken r at a time*, or an *r-combination*, is any subset of r elements. In other words, an r-combination is any selection of r of the n objects where order does not count.

Example 2.12: The combinations of the letters a, b, c, d taken 3 at a time are

$$\{a, b, c\},\ \{a, b, d\},\ \{a, c, d\},\ \{b, c, d\} \quad \text{or simply} \quad abc,\ abd,\ acd,\ bcd$$

Observe that the following combinations are equal:

$$abc,\ acb,\ bac,\ bca,\ cab,\ cba$$

That is, each denotes the same set $\{a, b, c\}$.

The number of combinations of n objects taken r at a time will be denoted by

$$C(n, r)$$

Before we give the general formula for $C(n, r)$, we consider a special case.

Example 2.13: We determine the number of combinations of the four letters a, b, c, d taken 3 at a time. Note that each combination consisting of three letters determines $3! = 6$ permutations of the letters in the combination:

Combinations	Permutations
abc	*abc, acb, bac, bca, cab, cba*
abd	*abd, adb, bad, bda, dab, dba*
acd	*acd, adc, cad, cda, dac, dca*
bcd	*bcd, bdc, cbd, cdb, dbc, dcb*

Thus the number of combinations multiplied by $3!$ equals the number of permutations:

$$C(4,3) \cdot 3! = P(4,3) \quad \text{or} \quad C(4,3) = \frac{P(4,3)}{3!}$$

Now $P(4,3) = 4 \cdot 3 \cdot 2 = 24$ and $3! = 6$; hence $C(4,3) = 4$ as noted above.

Since each combination of n objects taken r at a time determines $r!$ permutations of the objects, we can conclude that

$$P(n, r) = r!\,C(n, r)$$

Thus we obtain

Theorem 2.8: $C(n, r) = \dfrac{P(n,r)}{r!} = \dfrac{n!}{r!\,(n-r)!}$

Recall that the binomial coefficient $\binom{n}{r}$ was defined to be $\dfrac{n!}{r!\,(n-r)!}$; hence

$$\boxed{C(n, r) = \binom{n}{r}}$$

We shall use $C(n, r)$ and $\binom{n}{r}$ interchangeably.

Example 2.14: How many committees of 3 can be formed from 8 people? Each committee is essentially a combination of the 8 people taken 3 at a time. Thus

$$C(8,3) = \binom{8}{3} = \frac{8\cdot 7\cdot 6}{1\cdot 2\cdot 3} = 56$$

different committees can be formed.

ORDERED PARTITIONS

Suppose an urn A contains seven marbles numbered 1 through 7. We compute the number of ways we can draw, first, 2 marbles from the urn, then 3 marbles from the urn, and lastly 2 marbles from the urn. In other words, we want to compute the number of *ordered partitions*

$$(A_1, A_2, A_3)$$

of the set of 7 marbles into cells A_1 containing 2 marbles, A_2 containing 3 marbles and A_3 containing 2 marbles. We call these ordered partitions since we distinguish between

$$(\{1,2\}\ \{3,4,5\},\ \{6,7\}) \quad \text{and} \quad (\{6,7\},\ \{3,4,5\},\ \{1,2\})$$

each of which yields the same partition of A.

Since we begin with 7 marbles in the urn, there are $\binom{7}{2}$ ways of drawing the first 2 marbles, i.e. of determining the first cell A_1; following this, there are 5 marbles left in the urn and so there are $\binom{5}{3}$ ways of drawing the 3 marbles, i.e. of determining A_2; finally, there are 2 marbles left in the urn and so there are $\binom{2}{2}$ ways of determining the last cell A_3. Thus there are

$$\binom{7}{2}\binom{5}{3}\binom{2}{2} = \frac{7\cdot 6}{1\cdot 2}\cdot\frac{5\cdot 4\cdot 3}{1\cdot 2\cdot 3}\cdot\frac{2\cdot 1}{1\cdot 2} = 210$$

different ordered partitions of A into cells A_1 containing 2 marbles, A_2 containing 3 marbles, and A_3 containing 2 marbles.

Now observe that

$$\binom{7}{2}\binom{5}{3}\binom{2}{2} = \frac{7!}{2!\,5!}\cdot\frac{5!}{3!\,2!}\cdot\frac{2!}{2!\,0!} = \frac{7!}{2!\,3!\,2!}$$

since each numerator after the first is cancelled by the second term in the denominator of the previous factor. In a similar manner we prove (Problem 2.28)

Theorem 2.9: Let A contain n elements and let $n_1, n_2, \ldots, n_r$ be positive integers with $n_1 + n_2 + \cdots + n_r = n$. Then there exist

$$\frac{n!}{n_1!\,n_2!\,n_3!\cdots n_r!}$$

different ordered partitions of A of the form $(A_1, A_2, \ldots, A_r)$ where A_1 contains n_1 elements, A_2 contains n_2 elements, ..., and A_r contains n_r elements.

Example 2.15: In how many ways can 9 toys be divided between 4 children if the youngest child is to receive 3 toys and each of the other children 2 toys?

We wish to find the number of ordered partitions of the 9 toys into 4 cells containing 3, 2, 2 and 2 toys respectively. By the above theorem, there are

$$\frac{9!}{3!\,2!\,2!\,2!} = 7560$$

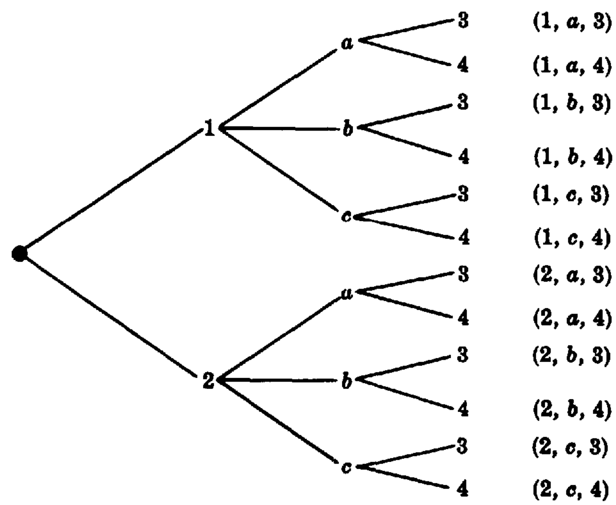
such ordered partitions.

TREE DIAGRAMS

A tree diagram is a device used to enumerate all the possible outcomes of a sequence of experiments where each experiment can occur in a finite number of ways. The construction of tree diagrams is illustrated in the following examples.

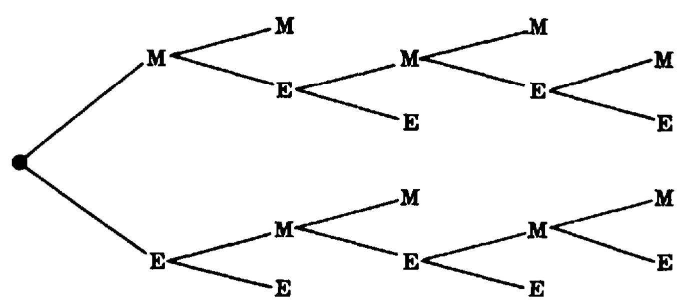
Example 2.16: Find the product set $A \times B \times C$ where $A = \{1,2\}$, $B = \{a,b,c\}$ and $C = \{3,4\}$.

The tree diagram follows:



Observe that the tree is constructed from left to right, and that the number of branches at each point corresponds to the number of possible outcomes of the next experiment.

Example 2.17: Mark and Eric are to play a tennis tournament. The first person to win two games in a row or who wins a total of three games wins the tournament. The following diagram shows the possible outcomes of the tournament.



Observe that there are 10 endpoints which correspond to the 10 possible outcomes of the tournament:

MM, MEMM, MEMEM, MEMEE, MEE, EMM, EMEMM, EMEME, EMEE, EE

The path from the beginning of the tree to the endpoint indicates who won which game in the individual tournament.

Solved Problems

FACTORIAL

2.1. Compute $4!$, $5!$, $6!$, $7!$ and $8!$.

$4! = 1\cdot 2\cdot 3\cdot 4 = 24$

$5! = 1\cdot 2\cdot 3\cdot 4\cdot 5 = 5\cdot 4! = 5\cdot 24 = 120$

$6! = 1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6 = 6\cdot 5! = 6\cdot 120 = 720$

$7! = 7\cdot 6! = 7\cdot 720 = 5040$

$8! = 8\cdot 7! = 8\cdot 5040 = 40{,}320$

2.2. Compute: (i) $\frac{13!}{11!}$, (ii) $\frac{7!}{10!}$.

(i) $\frac{13!}{11!} = \frac{13\cdot 12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} = 13\cdot 12 = 156$

or $\frac{13!}{11!} = \frac{13\cdot 12\cdot 11!}{11!} = 13\cdot 12 = 156$

(ii) $\frac{7!}{10!} = \frac{7!}{10\cdot 9\cdot 8\cdot 7!} = \frac{1}{10\cdot 9\cdot 8} = \frac{1}{720}$

2.3. Simplify: (i) $\frac{n!}{(n-1)!}$, (ii) $\frac{(n+2)!}{n!}$.

(i) $\frac{n!}{(n-1)!} = \frac{n(n-1)(n-2)\cdots 3\cdot 2\cdot 1}{(n-1)(n-2)\cdots 3\cdot 2\cdot 1} = n$ or, simply, $\frac{n!}{(n-1)!} = \frac{n(n-1)!}{(n-1)!} = n$

(ii) $\frac{(n+2)!}{n!} = \frac{(n+2)(n+1)n(n-1)(n-2)\cdots 3\cdot 2\cdot 1}{n(n-1)(n-2)\cdots 3\cdot 2\cdot 1} = (n+2)(n+1) = n^2+3n+2$

or, simply, $\frac{(n+2)!}{n!} = \frac{(n+2)(n+1)\cdot n!}{n!} = (n+2)(n+1) = n^2+3n+2$

PERMUTATIONS, ORDERED SAMPLES

2.4. If repetitions are not permitted, (i) how many 3 digit numbers can be formed from the six digits 2, 3, 5, 6, 7 and 9? (ii) How many of these are less than 400? (iii) How many are even? (iv) How many are odd? (v) How many are multiples of 5?

In each case draw three boxes [ ] [ ] [ ] to represent an arbitrary number, and then write in each box the number of digits that can be placed there.

(i) The box on the left can be filled in 6 ways; following this, the middle box can be filled in 5 ways; and, lastly, the box on the right can be filled in 4 ways: [6] [5] [4]. Thus there are $6\cdot 5\cdot 4 = 120$ numbers.

(ii) The box on the left can be filled in only 2 ways, by 2 or 3, since each number must be less than 400; the middle box can be filled in 5 ways; and, lastly, the box on the right can be filled in 4 ways: [2] [5] [4]. Thus there are $2\cdot 5\cdot 4 = 40$ numbers.

(iii) The box on the right can be filled in only 2 ways, by 2 or 6, since the numbers must be even; the box on the left can then be filled in 5 ways; and, lastly, the middle box can be filled in 4 ways: [5] [4] [2]. Thus there are $5\cdot 4\cdot 2 = 40$ numbers.

(iv) The box on the right can be filled in only 4 ways, by 3, 5, 7 or 9, since the numbers must be odd; the box on the left can then be filled in 5 ways; and, lastly, the box in the middle can be filled in 4 ways: [5] [4] [4]. Thus there are $5\cdot 4\cdot 4 = 80$ numbers.

(v) The box on the right can be filled in only 1 way, by 5, since the numbers must be multiples of 5; the box on the left can then be filled in 5 ways; and, lastly, the box in the middle can be filled in 4 ways: $\boxed{5}\ \boxed{4}\ \boxed{1}$. Thus there are $5\cdot 4\cdot 1 = 20$ numbers.

2.5. In how many ways can a party of 7 persons arrange themselves (i) in a row of 7 chairs? (ii) around a circular table?

(i) The seven persons can arrange themselves in a row in $7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 = 7!$ ways.

(ii) One person can sit at any place in the circular table. The other six persons can then arrange themselves in $6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 = 6!$ ways around the table.

This is an example of a *circular permutation*. In general, n objects can be arranged in a circle in $(n-1)(n-2)\cdots 3\cdot 2\cdot 1 = (n-1)!$ ways.

2.6. (i) In how many ways can 3 boys and 2 girls sit in a row? (ii) In how many ways can they sit in a row if the boys and girls are each to sit together? (iii) In how many ways can they sit in a row if just the girls are to sit together?

(i) The five persons can sit in a row in $5\cdot 4\cdot 3\cdot 2\cdot 1 = 5! = 120$ ways.

(ii) There are 2 ways to distribute them according to sex: BBBGG or GGBBB. In each case the boys can sit in $3\cdot 2\cdot 1 = 3! = 6$ ways, and the girls can sit in $2\cdot 1 = 2! = 2$ ways. Thus, altogether, there are $2\cdot 3!\cdot 2! = 2\cdot 6\cdot 2 = 24$ ways.

(iii) There are 4 ways to distribute them according to sex: GGBBB, BGGBB, BBGGB, BBBGG. Note that each way corresponds to the number, 0, 1, 2 or 3, of boys sitting to the left of the girls. In each case, the boys can sit in 3! ways, and the girls in 2! ways. Thus, altogether, there are $4\cdot 3!\cdot 2! = 4\cdot 6\cdot 2 = 48$ ways.

2.7. How many different signals, each consisting of 6 flags hung in a vertical line, can be formed from 4 identical red flags and 2 identical blue flags?

This problem concerns permutations with repetitions. There are $\frac{6!}{4!\,2!} = 15$ signals since there are 6 flags of which 4 are red and 2 are blue.

2.8. How many distinct permutations can be formed from all the letters of each word: (i) them, (ii) unusual, (iii) sociological?

(i) $4! = 24$, since there are 4 letters and no repetitions.

(ii) $\frac{7!}{3!} = 840$, since there are 7 letters of which 3 are u.

(iii) $\frac{12!}{3!\,2!\,2!\,2!}$, since there are 12 letters of which 3 are o, 2 are c, 2 are i, and 2 are l.

2.9. (i) In how many ways can 3 Americans, 4 Frenchmen, 4 Danes and 2 Italians be seated in a row so that those of the same nationality sit together?

(ii) Solve the same problem if they sit at a round table.

(i) The 4 nationalities can be arranged in a row in 4! ways. In each case the 3 Americans can be seated in 3! ways, the 4 Frenchmen in 4! ways, the 4 Danes in 4! ways, and the 2 Italians in 2! ways. Thus, altogether, there are $4!\,3!\,4!\,4!\,2! = 165{,}888$ arrangements.

(ii) The 4 nationalities can be arranged in a circle in 3! ways (see Problem 14.4 on circular permutations). In each case the 3 Americans can be seated in 3! ways, the 4 Frenchmen in 4! ways, the 4 Danes in 4! ways, and the 2 Italians in 2! ways. Thus, altogether, there are $3!\,3!\,4!\,4!\,2! = 41{,}472$ arrangements.

2.10. Suppose an urn contains 8 balls. Find the number of ordered samples of size 3 (i) with replacement, (ii) without replacement.

(i) Each ball in the ordered sample can be chosen in 8 ways; hence there are $8 \cdot 8 \cdot 8 = 8^3 = 512$ samples with replacement.

(ii) The first ball in the ordered sample can be chosen in 8 ways, the next in 7 ways, and the last in 6 ways. Thus there are $8 \cdot 7 \cdot 6 = 336$ samples without replacement.

2.11. Find n if (i) $P(n,2) = 72$, (ii) $P(n,4) = 42P(n,2)$, (iii) $2P(n,2) + 50 = P(2n,2)$.

(i) $P(n,2) = n(n-1) = n^2 - n$; hence $n^2 - n = 72$ or $n^2 - n - 72 = 0$ or $(n-9)(n+8) = 0$.

Since n must be positive, the only answer is $n = 9$.

(ii) $P(n,4) = n(n-1)(n-2)(n-3)$ and $P(n,2) = n(n-1)$. Hence

$$n(n-1)(n-2)(n-3) = 42n(n-1) \quad \text{or, if } n \neq 0, \neq 1, \quad (n-2)(n-3) = 42$$
$$\text{or} \quad n^2 - 5n + 6 = 42 \quad \text{or} \quad n^2 - 5n - 36 = 0 \quad \text{or} \quad (n-9)(n+4) = 0$$

Since n must be positive, the only answer is $n = 9$.

(iii) $P(n,2) = n(n-1) = n^2 - n$ and $P(2n,2) = 2n(2n-1) = 4n^2 - 2n$. Hence

$$2(n^2 - n) + 50 = 4n^2 - 2n \quad \text{or} \quad 2n^2 - 2n + 50 = 4n^2 - 2n \quad \text{or} \quad 50 = 2n^2 \quad \text{or} \quad n^2 = 25$$

Since n must be positive, the only answer is $n = 5$.

BINOMIAL COEFFICIENTS AND THEOREM

2.12. Compute: (i) $\binom{16}{3}$, (ii) $\binom{12}{4}$, (iii) $\binom{15}{5}$.

Recall that there are as many factors in the numerator as in the denominator.

(i) $\binom{16}{3} = \dfrac{16 \cdot 15 \cdot 14}{1 \cdot 2 \cdot 3} = 560$ (iii) $\binom{15}{5} = \dfrac{15 \cdot 14 \cdot 13 \cdot 12 \cdot 11}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 3003$

(ii) $\binom{12}{4} = \dfrac{12 \cdot 11 \cdot 10 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4} = 495$

2.13. Compute: (i) $\binom{8}{5}$, (ii) $\binom{9}{7}$, (iii) $\binom{10}{6}$.

(i) $\binom{8}{5} = \dfrac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 56$

Note that $8 - 5 = 3$; hence we could also compute $\binom{8}{5}$ as follows:

$$\binom{8}{5} = \binom{8}{3} = \frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3} = 56$$

(ii) Now $9 - 7 = 2$; hence $\binom{9}{7} = \binom{9}{2} = \dfrac{9 \cdot 8}{1 \cdot 2} = 36$.

(iii) Now $10 - 6 = 4$; hence $\binom{10}{6} = \binom{10}{4} = \dfrac{10 \cdot 9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4} = 210$.

2.14. Expand and simplify: $(2x + y^2)^5$.

$$\begin{aligned}(2x + y^2)^5 &= (2x)^5 + \frac{5}{1}(2x)^4(y^2) + \frac{5 \cdot 4}{1 \cdot 2}(2x)^3(y^2)^2 + \frac{5 \cdot 4}{1 \cdot 2}(2x)^2(y^2)^3 + \frac{5}{1}(2x)(y^2)^4 + (y^2)^5 \\ &= 32x^5 + 80x^4y^2 + 80x^3y^4 + 40x^2y^6 + 10xy^8 + y^{10}\end{aligned}$$

2.15. Expand and simplify: $(x^2 - 2y)^6$.

$$\begin{aligned}(x^2-2y)^6 &= (x^2)^6 + \frac{6}{1}(x^2)^5(-2y) + \frac{6\cdot 5}{1\cdot 2}(x^2)^4(-2y)^2 + \frac{6\cdot 5\cdot 4}{1\cdot 2\cdot 3}(x^2)^3(-2y)^3 \\ &\quad + \frac{6\cdot 5}{1\cdot 2}(x^2)^2(-2y)^4 + \frac{6}{1}(x^2)(-2y)^5 + (-2y)^6 \\ &= x^{12} - 12x^{10}y + 60x^8y^2 - 160x^6y^3 + 240x^4y^4 - 192x^2y^5 + 64y^6\end{aligned}$$

2.16. Prove: $2^4 = 16 = \binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4}$.

Expand $(1+1)^4$ using the binomial theorem:

$$\begin{aligned}2^4 = (1+1)^4 &= \binom{4}{0}1^4 + \binom{4}{1}1^3\,1^1 + \binom{4}{2}1^2\,1^2 + \binom{4}{3}1^1\,1^3 + \binom{4}{4}1^4 \\ &= \binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4}\end{aligned}$$

2.17. Prove Theorem 2.6: $\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$.

Now $\binom{n}{r-1} + \binom{n}{r} = \dfrac{n!}{(r-1)!\cdot(n-r+1)!} + \dfrac{n!}{r!\cdot(n-r)!}$. To obtain the same denominator in both fractions, multiply the first fraction by $\dfrac{r}{r}$ and the second fraction by $\dfrac{n-r+1}{n-r+1}$. Hence

$$\begin{aligned}\binom{n}{r-1} + \binom{n}{r} &= \frac{r\cdot n!}{r\cdot(r-1)!\cdot(n-r+1)!} + \frac{(n-r+1)\cdot n!}{r!\cdot(n-r+1)\cdot(n-r)!} \\ &= \frac{r\cdot n!}{r!\,(n-r+1)!} + \frac{(n-r+1)\cdot n!}{r!\,(n-r+1)!} \\ &= \frac{r\cdot n! + (n-r+1)\cdot n!}{r!\,(n-r+1)!} = \frac{[r+(n-r+1)]\cdot n!}{r!\,(n-r+1)!} \\ &= \frac{(n+1)n!}{r!\,(n-r+1)!} = \frac{(n+1)!}{r!\,(n-r+1)!} = \binom{n+1}{r}\end{aligned}$$

2.18. Prove the Binomial Theorem 2.5: $(a+b)^n = \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r$.

The theorem is true for $n = 1$, since

$$\sum_{r=0}^{1}\binom{1}{r}a^{1-r}b^r = \binom{1}{0}a^1b^0 + \binom{1}{1}a^0b^1 = a + b = (a+b)^1$$

We assume the theorem holds for $(a+b)^n$ and prove it is true for $(a+b)^{n+1}$.

$$\begin{aligned}(a+b)^{n+1} &= (a+b)(a+b)^n \\ &= (a+b)\left[a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{r-1}a^{n-r+1}b^{r-1}\right. \\ &\quad \left. + \binom{n}{r}a^{n-r}b^r + \cdots + \binom{n}{1}ab^{n-1} + b^n\right]\end{aligned}$$

Now the term in the product which contains b^r is obtained from

$$\begin{aligned}b\left[\binom{n}{r-1}a^{n-r+1}b^{r-1}\right] + a\left[\binom{n}{r}a^{n-r}b^r\right] &= \binom{n}{r-1}a^{n-r+1}b^r + \binom{n}{r}a^{n-r+1}b^r \\ &= \left[\binom{n}{r-1} + \binom{n}{r}\right]a^{n-r+1}b^r\end{aligned}$$

But, by Theorem 2.6, $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$. Thus the term containing b^r is $\binom{n+1}{r} a^{n-r+1} b^r$. Note that $(a+b)(a+b)^n$ is a polynomial of degree $n+1$ in b. Consequently,

$$(a+b)^{n+1} = (a+b)(a+b)^n = \sum_{r=0}^{n+1} \binom{n+1}{r} a^{n-r+1} b^r$$

which was to be proved.

2.19. Compute the following multinomial coefficients:

$$\text{(i)} \binom{6}{3,2,1}, \quad \text{(ii)} \binom{8}{4,2,2,0}, \quad \text{(iii)} \binom{10}{5,3,2,2}$$

(i) $\binom{6}{3,2,1} = \frac{6!}{3!\,2!\,1!} = \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{3\cdot2\cdot1\cdot2\cdot1\cdot1} = 60$

(ii) $\binom{8}{4,2,2,0} = \frac{8!}{4!\,2!\,2!\,0!} = \frac{8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{4\cdot3\cdot2\cdot1\cdot2\cdot1\cdot2\cdot1\cdot1} = 420$

(iii) The expression $\binom{10}{5,3,2,2}$ has no meaning since $5+3+2+2 \neq 10$.

COMBINATIONS

2.20. In how many ways can a committee consisting of 3 men and 2 women be chosen from 7 men and 5 women?

The 3 men can be chosen from the 7 men in $\binom{7}{3}$ ways, and the 2 women can be chosen from the 5 women in $\binom{5}{2}$ ways. Hence the committee can be chosen in $\binom{7}{3}\binom{5}{2} = \frac{7\cdot6\cdot5}{1\cdot2\cdot3}\cdot\frac{5\cdot4}{1\cdot2} = 350$ ways.

2.21. A delegation of 4 students is selected each year from a college to attend the National Student Association annual meeting. (i) In how many ways can the delegation be chosen if there are 12 eligible students? (ii) In how many ways if two of the eligible students will not attend the meeting together? (iii) In how many ways if two of the eligible students are married and will only attend the meeting together?

(i) The 4 students can be chosen from the 12 students in $\binom{12}{4} = \frac{12\cdot11\cdot10\cdot9}{1\cdot2\cdot3\cdot4} = 495$ ways.

(ii) Let A and B denote the students who will not attend the meeting together.

Method 1.

If neither A nor B is included, then the delegation can be chosen in $\binom{10}{4} = \frac{10\cdot9\cdot8\cdot7}{1\cdot2\cdot3\cdot4} =$ 210 ways. If either A or B, but not both, is included, then the delegation can be chosen in $2\cdot\binom{10}{3} = 2\cdot\frac{10\cdot9\cdot8}{1\cdot2\cdot3} = 240$ ways. Thus, altogether, the delegation can be chosen in $210 + 240 = 450$ ways.

Method 2.

If A and B are both included, then the other 2 members of the delegation can be chosen in $\binom{10}{2} = 45$ ways. Thus there are $495 - 45 = 450$ ways the delegation can be chosen if A and B are not both included.

(iii) Let C and D denote the married students. If C and D do not go, then the delegation can be chosen in $\binom{10}{4} = 210$ ways. If both C and D go, then the delegation can be chosen in $\binom{10}{2} = 45$ ways. Altogether, the delegation can be chosen in $210 + 45 = 255$ ways.

2.22. A student is to answer 8 out of 10 questions on an exam. (i) How many choices has he? (ii) How many if he must answer the first 3 questions? (iii) How many if he must answer at least 4 of the first 5 questions?

(i) The 8 questions can be selected in $\binom{10}{8} = \binom{10}{2} = \frac{10 \cdot 9}{1 \cdot 2} = 45$ ways.

(ii) If he answers the first 3 questions, then he can choose the other 5 questions from the last 7 questions in $\binom{7}{5} = \binom{7}{2} = \frac{7 \cdot 6}{1 \cdot 2} = 21$ ways.

(iii) If he answers all the first 5 questions, then he can choose the other 3 questions from the last 5 in $\binom{5}{3} = 10$ ways. On the other hand, if he answers only 4 of the first 5 questions, then he can choose these 4 in $\binom{5}{4} = \binom{5}{1} = 5$ ways, and he can choose the other 4 questions from the last 5 in $\binom{5}{4} = \binom{5}{1} = 5$ ways; hence he can choose the 8 questions in $5 \cdot 5 = 25$ ways. Thus he has a total of 35 choices.

2.23. Find the number of subsets of a set X containing n elements.

Method 1.

The number of subsets of X with $r \leq n$ elements is given by $\binom{n}{r}$. Hence, altogether, there are

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n}$$

subsets of X. The above sum (Problem 2.51) is equal to 2^n, i.e. there are 2^n subsets of X.

Method 2.

There are two possibilities for each element of X: either it belongs to the subset or it doesn't; hence there are

$$\overbrace{2 \cdot 2 \cdot \cdots \cdot 2}^{n \text{ times}} = 2^n$$

ways to form a subset of X, i.e. there are 2^n different subsets of X.

2.24. In how many ways can a teacher choose one or more students from six eligible students?

Method 1.

By the preceding problem, there are $2^6 = 64$ subsets of the set consisting of the six students. However, the empty set must be deleted since one or more students are chosen. Accordingly there are $2^6 - 1 = 64 - 1 = 63$ ways to choose the students.

Method 2.

Either 1, 2, 3, 4, 5 or 6 students are chosen. Hence the number of choices is

$$\binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} + \binom{6}{5} + \binom{6}{6} = 6 + 15 + 20 + 15 + 6 + 1 = 63$$

ORDERED AND UNORDERED PARTITIONS

2.25. In how many ways can 7 toys be divided among 3 children if the youngest gets 3 toys and each of the others gets 2?

We seek the number of ordered partitions of 7 objects into cells containing 3, 2 and 2 objects, respectively. By Theorem 2.9, there are $\frac{7!}{3!\,2!\,2!} = 210$ such partitions.

2.26. There are 12 students in a class. In how many ways can the 12 students take 3 different tests if 4 students are to take each test?

Method 1.

We seek the number of ordered partitions of the 12 students into cells containing 4 students each. By Theorem 2.9, there are $\frac{12!}{4!\,4!\,4!} = 34{,}650$ such partitions.

Method 2.

There are $\binom{12}{4}$ ways to choose 4 students to take the first test; following this, there are $\binom{8}{4}$ ways to choose 4 students to take the second test. The remaining students take the third test. Thus, altogether, there are $\binom{12}{4}\cdot\binom{8}{4} = 495\cdot 70 = 34{,}650$ ways for the students to take the tests.

2.27. In how many ways can 12 students be partitioned into 3 teams, A_1, A_2 and A_3, so that each team contains 4 students?

Method 1.

Observe that each partition $\{A_1, A_2, A_3\}$ of the students can be arranged in $3! = 6$ ways as an ordered partition. Since (see the preceding problem) there are $\frac{12!}{4!\,4!\,4!} = 34{,}650$ such ordered partitions, there are $34{,}650/6 = 5775$ (unordered) partitions.

Method 2.

Let A denote one of the students. Then there are $\binom{11}{3}$ ways to choose 3 other students to be on the same team as A. Now let B denote a student who is not on the same team as A; then there are $\binom{7}{3}$ ways to choose 3 students of the remaining students to be on the same team as B. The remaining 4 students constitute the third team. Thus, altogether, there are $\binom{11}{3}\cdot\binom{7}{3} = 165\cdot 35 = 5775$ ways to partition the students.

2.28. Prove Theorem 2.9: Let A contain n elements and let $n_1, n_2, \ldots, n_r$ be positive integers with $n_1 + n_2 + \cdots + n_r = n$. Then there exist

$$\frac{n!}{n_1!\,n_2!\,n_3!\cdots n_r!}$$

different ordered partitions of A of the form $(A_1, A_2, \ldots, A_r)$ where A_1 contains n_1 elements, A_2 contains n_2 elements, $\ldots$, and A_r contains n_r elements.

We begin with n elements in A; hence there are $\binom{n}{n_1}$ ways of selecting the cell A_1. Following this, there are $n - n_1$ elements left, i.e. in $A \setminus A_1$, and so there are $\binom{n-n_1}{n_2}$ ways of selecting A_2. Similarly, for $i = 3, \ldots, r$, there are $\binom{n-n_1-\cdots-n_{i-1}}{n_i}$ ways of selecting A_i. Thus there are

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-\cdots-n_{r-1}}{n_r} \qquad (*)$$

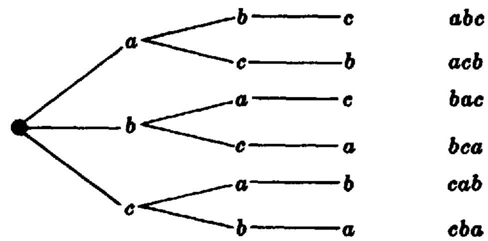
different ordered partitions of A. Now (*) is equal to

$$\frac{n!}{n_1!\,(n-n_1)!}\cdot\frac{(n-n_1)!}{n_2!\,(n-n_1-n_2)!}\cdot\cdots\cdot\frac{(n-n_1-\cdots-n_{r-1})!}{n_r!\,(n-n_1-\cdots-n_r)!}$$

But this is equal to $\frac{n!}{n_1!\,n_2!\cdots n_r!}$ since each numerator after the first is cancelled by the second term in the denominator and since $(n-n_1-\cdots-n_r)! = 0! = 1$. Thus the theorem is proved.

TREE DIAGRAMS

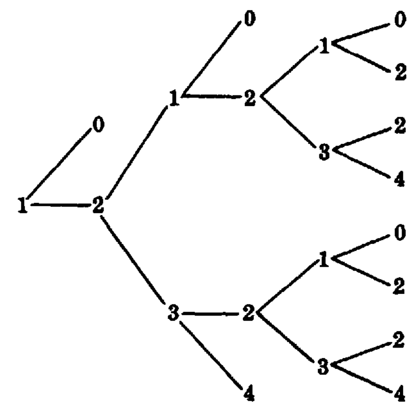
2.29. Construct the tree diagram for the number of permutations of $\{a, b, c\}$.



The six permutations are listed on the right of the diagram.

2.30. A man has time to play roulette at most five times. At each play he wins or loses a dollar. The man begins with one dollar and will stop playing before the five times if he loses all his money or if he wins three dollars, i.e. if he has four dollars. Find the number of ways that the betting can occur.

The tree diagram on the right describes the way the betting can occur. Each number in the diagram denotes the number of dollars the man has at that point. Observe that the betting can occur in 11 different ways. Note that he will stop betting before the five times are up in only three of the cases.



Supplementary Problems

FACTORIAL

2.31. Compute: (i) 9!, (ii) 10!, (iii) 11!

2.32. Compute: (i) $\frac{16!}{14!}$, (ii) $\frac{14!}{11!}$, (iii) $\frac{8!}{10!}$, (iv) $\frac{10!}{13!}$.

2.33. Simplify: (i) $\frac{(n+1)!}{n!}$, (ii) $\frac{n!}{(n-2)!}$, (iii) $\frac{(n-1)!}{(n+2)!}$, (iv) $\frac{(n-r+1)!}{(n-r-1)!}$.

PERMUTATIONS

2.34. (i) How many automobile license plates can be made if each plate contains 2 different letters followed by 3 different digits? (ii) Solve the problem if the first digit cannot be 0.

2.35. There are 6 roads between A and B and 4 roads between B and C.

(i) In how many ways can one drive from A to C by way of B?

(ii) In how many ways can one drive roundtrip from A to C by way of B?

(iii) In how many ways can one drive roundtrip from A to C without using the same road more than once?

2.36. Find the number of ways in which 6 people can ride a toboggan if one of three must drive.

2.37. (i) Find the number of ways in which five persons can sit in a row.
(ii) How many ways are there if two of the persons insist on sitting next to one another?

2.38. Solve the preceding problem if they sit around a circular table.

2.39. (i) Find the number of four letter words that can be formed from the letters of the word HISTORY. (ii) How many of them contain only consonants? (iii) How many of them begin and end in a consonant? (iv) How many of them begin with a vowel? (v) How many contain the letter Y? (vi) How many begin with T and end in a vowel? (vii) How many begin with T and also contain S? (viii) How many contain both vowels?

2.40. How many different signals, each consisting of 8 flags hung in a vertical line, can be formed from 4 red flags, 2 blue flags and 2 green flags?

2.41. Find the number of permutations that can be formed from all the letters of each word: (i) queue, (ii) committee, (iii) proposition, (iv) baseball.

2.42. (i) Find the number of ways in which 4 boys and 4 girls can be seated in a row if the boys and girls are to have alternate seats.
(ii) Find the number of ways if they sit alternately and if one boy and one girl are to sit in adjacent seats.
(iii) Find the number of ways if they sit alternately and if one boy and one girl must not sit in adjacent seats.

2.43. Solve the preceding problem if they sit around a circular table.

2.44. An urn contains 10 balls. Find the number of ordered samples (i) of size 3 with replacement, (ii) of size 3 without replacement, (iii) of size 4 with replacement, (iv) of size 5 without replacement.

2.45. Find the number of ways in which 5 large books, 4 medium-size books and 3 small books can be placed on a shelf so that all books of the same size are together.

2.46. Consider all positive integers with 3 different digits. (Note that 0 cannot be the first digit.) (i) How many are greater than 700? (ii) How many are odd? (iii) How many are even? (iv) How many are divisible by 5?

2.47. (i) Find the number of distinct permutations that can be formed from all of the letters of the word ELEVEN. (ii) How many of them begin and end with E? (iii) How many of them have the 3 E's together? (iv) How many begin with E and end with N?

BINOMIAL COEFFICIENTS AND THEOREM

2.48. Compute: (i) $\binom{5}{2}$, (ii) $\binom{7}{3}$, (iii) $\binom{14}{2}$, (iv) $\binom{6}{4}$, (v) $\binom{20}{17}$, (vi) $\binom{18}{15}$.

2.49. Compute: (i) $\binom{9}{3,\,5,\,1}$, (ii) $\binom{7}{3,\,2,\,2,\,0}$, (iii) $\binom{6}{2,\,2,\,1,\,1,\,0}$.

2.50. Expand and simplify: (i) $(2x+y^2)^3$, (ii) $(x^2-3y)^4$, (iii) $(\frac{1}{2}a+2b)^5$, (iv) $(2a^2-b)^6$.

2.51. Show that $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n} = 2^n$.

2.52. Show that $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots \pm \binom{n}{n} = 0$.

2.53. Find the term in the expansion of $(2x^2-\frac{1}{2}y^3)^8$ which contains x^8.

2.54. Find the term in the expansion of $(3xy^2-z^2)^7$ which contains y^6.

COMBINATIONS

2.55. A class contains 9 boys and 3 girls. (i) In how many ways can the teacher choose a committee of 4? (ii) How many of them will contain at least one girl? (iii) How many of them will contain exactly one girl?

2.56. A woman has 11 close friends. (i) In how many ways can she invite 5 of them to dinner? (ii) In how many ways if two of the friends are married and will not attend separately? (iii) In how many ways if two of them are not on speaking terms and will not attend together?

2.57. There are 10 points $A, B, \ldots$ in a plane, no three on the same line. (i) How many lines are determined by the points? (ii) How many of these lines do not pass through A or B? (iii) How many triangles are determined by the points? (iv) How many of these triangles contain the point A? (v) How many of these triangles contain the side AB?

2.58. A student is to answer 10 out of 13 questions on an exam. (i) How many choices has he? (ii) How many if he must answer the first two questions? (iii) How many if he must answer the first or second question but not both? (iv) How many if he must answer exactly 3 of the first 5 questions? (v) How many if he must answer at least 3 of the first 5 questions?

2.59. A man is dealt a poker hand (5 cards) from an ordinary playing deck. In how many ways can he be dealt (i) a straight flush, (ii) four of a kind, (iii) a straight, (iv) a pair of aces, (v) two of a kind (a pair)?

2.60. The English alphabet has 26 letters of which 5 are vowels.

(i) How many 5 letter words containing 3 different consonants and 2 different vowels can be formed?
(ii) How many of them contain the letter b?
(iii) How many of them contain the letters b and c?
(iv) How many of them begin with b and contain the letter c?
(v) How many of them begin with b and end with c?
(vi) How many of them contain the letters a and b?
(vii) How many of them begin with a and contain b?
(viii) How many of them begin with b and contain a?
(ix) How many of them begin with a and end with b?
(x) How many of them contain the letters a, b and c?

ORDERED AND UNORDERED PARTITIONS

2.61. In how many ways can 9 toys be divided evenly among 3 children?

2.62. In how many ways can 9 students be evenly divided into three teams?

2.63. In how many ways can 10 students be divided into three teams, one containing 4 students and the others 3?

2.64. There are 12 balls in an urn. In how many ways can 3 balls be drawn from the urn, four times in succession, all without replacement?

2.65. In how many ways can a club with 12 members be partitioned into three committees containing 5, 4 and 3 members respectively?

2.66. In how many ways can n students be partitioned into two teams containing at least one student?

2.67. In how many ways can 14 men be partitioned into 6 committees where 2 of the committees contain 3 men and the others 2?

TREE DIAGRAMS

2.68. Construct the tree diagram for the number of permutations of $\{a, b, c, d\}$.

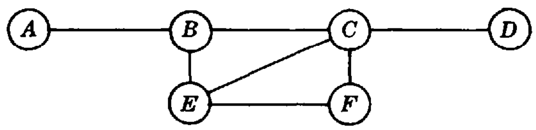
2.69. Find the product set $\{1, 2, 3\} \times \{2, 4\} \times \{2, 3, 4\}$ by constructing the appropriate tree diagram.

2.70. Teams A and B play in a basketball tournament. The first team that wins two games in a row or a total of four games wins the tournament. Find the number of ways the tournament can occur.

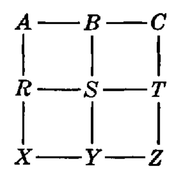
2.71. A man has time to play roulette five times. He wins or loses a dollar at each play. The man begins with two dollars and will stop playing before the five times if he loses all his money or wins three dollars (i.e. has five dollars). Find the number of ways the playing can occur.

2.72. A man is at the origin on the x-axis and takes a unit step either to the left or to the right. He stops after 5 steps or if he reaches 3 or -2. Construct the tree diagram to describe all possible paths the man can travel.

2.73. In the following diagram let $A, B, \ldots, F$ denote islands, and the lines connecting them bridges. A man begins at A and walks from island to island. He stops for lunch when he cannot continue to walk without crossing the same bridge twice. Find the number of ways that he can take his walk before eating lunch.



2.74. Consider the adjacent diagram with nine points $A, B, C, R, S, T, X, Y, Z$. A man begins at X and is allowed to move horizontally or vertically, one step at a time. He stops when he cannot continue to walk without reaching the same point more than once. Find the number of ways he can take his walk, if he first moves from X to R. (By symmetry, the total number of ways is twice this.)



Answers to Supplementary Problems

2.31. (i) 362,880 (ii) 3,628,800 (iii) 39,916,800

2.32. (i) 240 (ii) 2184 (iii) 1/90 (iv) 1/1716

2.33. (i) $n+1$ (ii) $n(n-1) = n^2 - n$ (iii) $1/[n(n+1)(n+2)]$ (iv) $(n-r)(n-r+1)$

2.34. (i) $26 \cdot 25 \cdot 10 \cdot 9 \cdot 8 = 468{,}000$ (ii) $26 \cdot 25 \cdot 9 \cdot 9 \cdot 8 = 421{,}200$

2.35. (i) $6 \cdot 4 = 24$ (ii) $6 \cdot 4 \cdot 4 \cdot 6 = 24 \cdot 24 = 576$ (iii) $6 \cdot 4 \cdot 3 \cdot 5 = 360$

2.36. $3 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 360$

2.37. (i) $5! = 120$ (ii) $4 \cdot 2! \cdot 3! = 48$

2.38. (i) $4! = 24$ (ii) $2!\,3! = 12$

2.39. (i) $7 \cdot 6 \cdot 5 \cdot 4 = 840$ (iii) $5 \cdot 5 \cdot 4 \cdot 4 = 400$ (v) $4 \cdot 6 \cdot 5 \cdot 4 = 480$ (vii) $1 \cdot 3 \cdot 5 \cdot 4 = 60$
(ii) $5 \cdot 4 \cdot 3 \cdot 2 = 120$ (iv) $2 \cdot 6 \cdot 5 \cdot 4 = 240$ (vi) $1 \cdot 5 \cdot 4 \cdot 2 = 40$ (viii) $4 \cdot 3 \cdot 5 \cdot 4 = 240$

2.40. $\frac{8!}{4!\,2!\,2!} = 420$

2.41. (i) $\frac{5!}{2!\,2!} = 30$ (ii) $\frac{9!}{2!\,2!\,2!} = 45{,}360$ (iii) $\frac{11!}{2!\,3!\,2!} = 1{,}663{,}200$ (iv) $\frac{8!}{2!\,2!\,2!} = 5040$

2.42. (i) $2 \cdot 4! \cdot 4! = 1152$ (ii) $2 \cdot 7 \cdot 3! \cdot 3! = 504$ (iii) $1152 - 504 = 648$

2.43. (i) $3! \cdot 4! = 144$ (ii) $2 \cdot 3! \cdot 3! = 72$ (iii) $144 - 72 = 72$

2.44. (i) $10 \cdot 10 \cdot 10 = 1000$ (iii) $10 \cdot 10 \cdot 10 \cdot 10 = 10{,}000$
(ii) $10 \cdot 9 \cdot 8 = 720$ (iv) $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 = 30{,}240$

2.45. $3!\,5!\,4!\,3! = 103{,}680$

2.46. (i) $3 \cdot 9 \cdot 8 = 216$ (ii) $8 \cdot 8 \cdot 5 = 320$
(iii) $9 \cdot 8 \cdot 1 = 72$ end in 0, and $8 \cdot 8 \cdot 4 = 256$ end in the other even digits; hence, altogether, $72 + 256 = 328$ are even.
(iv) $9 \cdot 8 \cdot 1 = 72$ end in 0, and $8 \cdot 8 \cdot 1 = 64$ end in 5; hence, altogether, $72 + 64 = 136$ are divisible by 5.

2.47. (i) $\frac{6!}{3!} = 120$ (ii) $4! = 24$ (iii) $4 \cdot 3! = 24$ (iv) $\frac{4!}{2!} = 12$

2.48. (i) 10 (ii) 35 (iii) 91 (iv) 15 (v) 1140 (vi) 816

2.49. (i) 504 (ii) 210 (iii) 180

2.50. (i) $8x^3 + 12x^2y^2 + 6xy^4 + y^6$
(ii) $x^8 - 12x^6y + 54x^4y^2 - 108x^2y^3 + 81y^4$
(iii) $a^5/32 + 5a^4b/8 + 5a^3b^2 + 20a^2b^3 + 40ab^4 + 32b^5$
(iv) $64a^{12} - 192a^{10}b + 240a^8b^2 - 160a^6b^3 + 60a^4b^4 - 12a^2b^5 + b^6$

2.51. *Hint.* Expand $(1+1)^n$. **2.53.** $70x^8y^{12}$

2.52. *Hint.* Expand $(1-1)^n$. **2.54.** $945x^3y^6z^8$

2.55. (i) $\binom{12}{4} = 495$, (ii) $\binom{12}{4} - \binom{9}{4} = 369$, (iii) $3 \cdot \binom{9}{3} = 252$

2.56. (i) $\binom{11}{5} = 462$, (ii) $\binom{9}{3} + \binom{9}{5} = 210$, (iii) $\binom{9}{5} + 2 \cdot \binom{9}{4} = 378$

2.57. (i) $\binom{10}{2} = 45$, (ii) $\binom{8}{2} = 28$, (iii) $\binom{10}{3} = 120$, (iv) $\binom{9}{2} = 36$, (v) 8

2.58. (i) $\binom{13}{10} = \binom{13}{3} = 286$
(ii) $\binom{11}{8} = \binom{11}{3} = 165$
(iii) $2 \cdot \binom{11}{9} = 2 \cdot \binom{11}{2} = 110$
(iv) $\binom{5}{3}\binom{8}{7} = 80$
(v) $\binom{5}{3}\binom{8}{7} + \binom{5}{4}\binom{8}{6} + \binom{5}{5}\binom{8}{5} = 276$

2.59. (i) $4 \cdot 10 = 40$, (ii) $13 \cdot 48 = 624$, (iii) $10 \cdot 4^5 - 40 = 10{,}200$. (We subtract the number of straight flushes.) (iv) $\binom{4}{2}\binom{12}{3} \cdot 4^3 = 84{,}480$, (v) $13 \cdot \binom{4}{2}\binom{12}{3} \cdot 4^3 = 1{,}098{,}240$

2.60. (i) $\binom{21}{3}\binom{5}{2} \cdot 5! = 1{,}596{,}000$
(ii) $\binom{20}{2}\binom{5}{2} \cdot 5! = 228{,}000$
(iii) $19 \cdot \binom{5}{2} \cdot 5! = 22{,}800$
(iv) $19 \cdot \binom{5}{2} \cdot 4! = 4560$
(v) $19 \cdot \binom{5}{2} \cdot 3! = 1140$
(vi) $4 \cdot \binom{20}{2} \cdot 5! = 91{,}200$
(vii) $4 \cdot \binom{20}{2} \cdot 4! = 18{,}240$
(viii) 18,240 (same as (vii))
(ix) $4 \cdot \binom{20}{2} \cdot 3! = 4560$
(x) $4 \cdot 19 \cdot 5! = 9120$

2.61. $\dfrac{9!}{3!\,3!\,3!} = 1680$

2.62. $1680/3! = 280$ or $\dbinom{8}{2}\dbinom{5}{2} = 280$

2.63. $\dfrac{10!}{4!\,3!\,3!} \cdot \dfrac{1}{2!} = 2100$ or $\dbinom{10}{4}\dbinom{5}{2} = 2100$

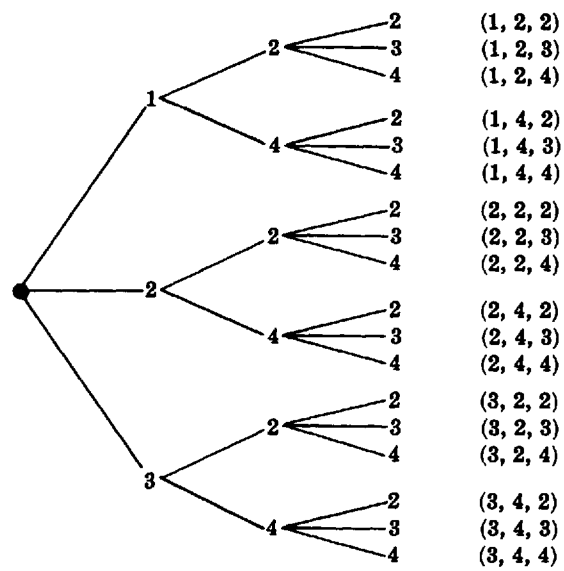
2.64. $\dfrac{12!}{3!\,3!\,3!\,3!} = 369{,}600$

2.65. $\dfrac{12!}{5!\,4!\,3!} = 27{,}720$

2.66. $2^{n-1} - 1$

2.67. $\dfrac{14!}{3!\,3!\,2!\,2!\,2!\,2!} \cdot \dfrac{1}{2!\,4!} = 3{,}153{,}150$

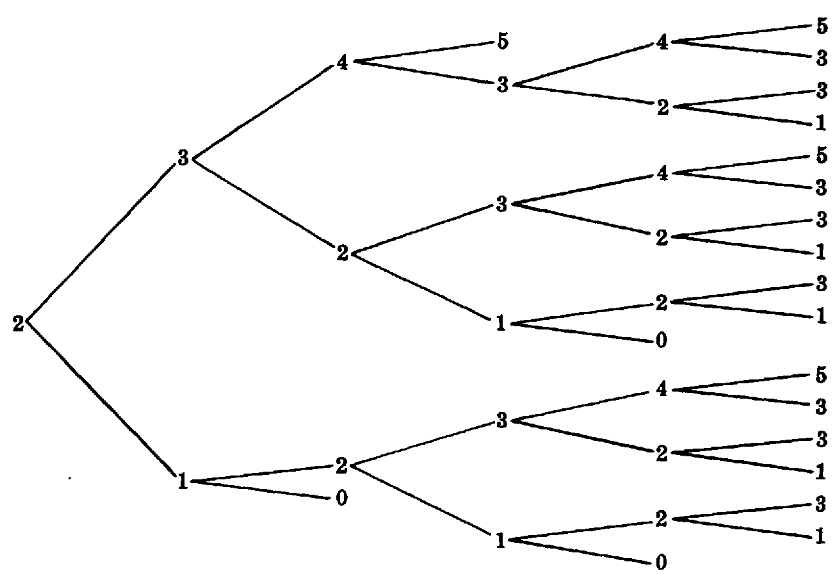
2.69.



The eighteen elements of the product set are listed to the right of the tree diagram.

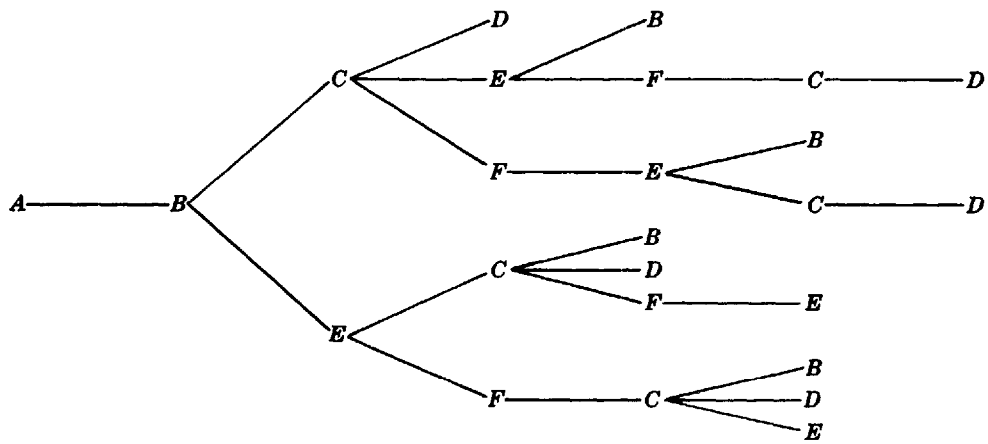
2.70. 14 ways

2.71. 20 ways (as seen in the following diagram):



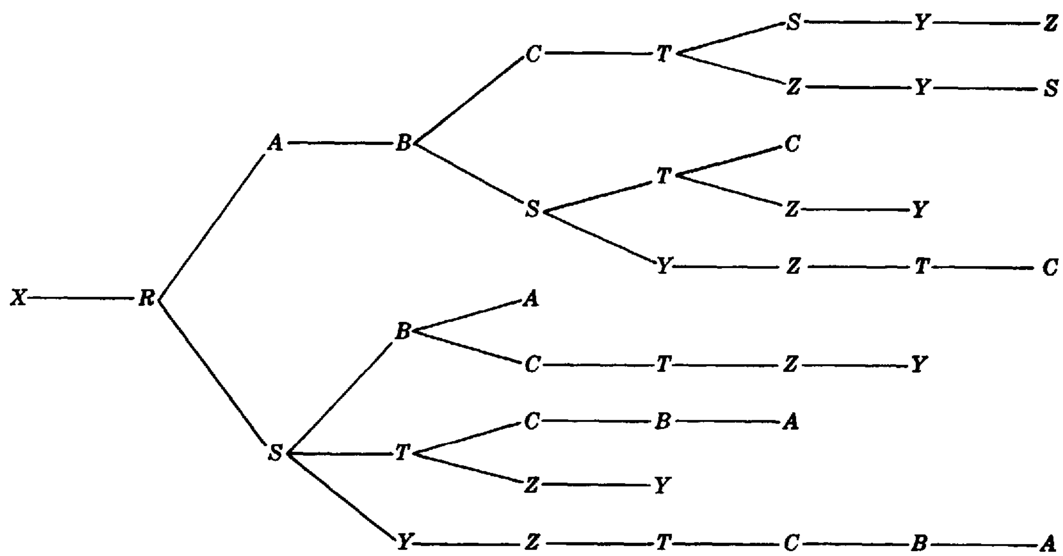
2.72. *Hint.* The tree is essentially the same as the tree of the preceding problem.

2.73. The appropriate tree diagram follows:



There are eleven ways to take his walk. Observe that he must eat his lunch at either B, D or E.

2.74. The appropriate tree diagram follows:



There are 10 different trips. (Note that in only 4 of them are all nine points covered.)

Chapter 3

Introduction to Probability

INTRODUCTION

Probability is the study of random or nondeterministic experiments. If a die is tossed in the air, then it is certain that the die will come down, but it is not certain that, say, a 6 will appear. However, suppose we repeat this experiment of tossing a die; let s be the number of successes, i.e. the number of times a 6 appears, and let n be the number of tosses. Then it has been empirically observed that the ratio $f = s/n$, called the *relative frequency*, becomes stable in the long run, i.e. approaches a limit. This stability is the basis of probability theory.

In probability theory, we define a mathematical model of the above phenomenon by assigning "probabilities" (or: the limit values of the relative frequencies) to the "events" connected with an experiment. Naturally, the reliability of our mathematical model for a given experiment depends upon the closeness of the assigned probabilities to the actual relative frequency. This then gives rise to problems of testing and reliability which form the subject matter of statistics.

Historically, probability theory began with the study of games of chance, such as roulette and cards. The probability p of an event A was defined as follows: if A can occur in s ways out of a total of n equally likely ways, then

$$p = P(A) = \frac{s}{n}$$

For example, in tossing a die an even number can occur in 3 ways out of 6 "equally likely" ways; hence $p = \frac{3}{6} = \frac{1}{2}$. This classical definition of probability is essentially circular since the idea of "equally likely" is the same as that of "with equal probability" which has not been defined. The modern treatment of probability theory is purely axiomatic. This means that the probabilities of our events can be perfectly arbitrary, except that they must satisfy certain axioms listed below. The classical theory will correspond to the special case of so-called *equiprobable spaces.*

SAMPLE SPACE AND EVENTS

The set S of all possible outcomes of some given experiment is called the *sample space.* A particular outcome, i.e. an element in S, is called a *sample point* or *sample.* An *event* A is a set of outcomes or, in other words, a subset of the sample space S. The event $\{a\}$ consisting of a single sample $a \in S$ is called an *elementary event.* The empty set $\emptyset$ and S itself are events; $\emptyset$ is sometimes called the *impossible* event, and S the *certain* or *sure* event.

We can combine events to form new events using the various set operations:

(i) $A \cup B$ is the event that occurs iff A occurs *or* B occurs (or both);

(ii) $A \cap B$ is the event that occurs iff A occurs *and* B occurs;

(iii) A^c, the complement of A, is the event that occurs iff A does *not* occur.

Two events A and B are called *mutually exclusive* if they are disjoint, i.e. if $A \cap B = \emptyset$. In other words, A and B are mutually exclusive if they cannot occur simultaneously.

Example 3.1: Experiment: Toss a die and observe the number that appears on top. Then the sample space consists of the six possible numbers:

$$S = \{1, 2, 3, 4, 5, 6\}$$

Let A be the event that an even number occurs, B that an odd number occurs and C that a prime number occurs:

$$A = \{2, 4, 6\}, \quad B = \{1, 3, 5\}, \quad C = \{2, 3, 5\}$$

Then:

$A \cup C = \{2, 3, 4, 5, 6\}$ is the event that an even or a prime number occurs;

$B \cap C = \{3, 5\}$ is the event that an odd prime number occurs;

$C^c = \{1, 4, 6\}$ is the event that a prime number does not occur.

Note that A and B are mutually exclusive: $A \cap B = \emptyset$; in other words, an even number and an odd number cannot occur simultaneously.

Example 3.2: Experiment: Toss a coin 3 times and observe the sequence of heads (H) and tails (T) that appears. The sample space S consists of eight elements:

$$S = \{\text{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}\}$$

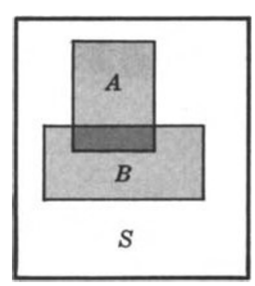
Let A be the event that two or more heads appear consecutively, and B that all the tosses are the same:

$$A = \{\text{HHH, HHT, THH}\} \quad \text{and} \quad B = \{\text{HHH, TTT}\}$$

Then $A \cap B = \{\text{HHH}\}$ is the elementary event in which only heads appear. The event that 5 heads appear is the empty set $\emptyset$.

Example 3.3: Experiment: Toss a coin until a head appears and then count the number of times the coin was tossed. The sample space of this experiment is $S = \{1, 2, 3, \ldots, \infty\}$. Here ∞ refers to the case when a head never appears and so the coin is tossed an infinite number of times. This is an example of a sample space which is *countably infinite*.

Example 3.4: Experiment: Let a pencil drop, head first, into a rectangular box and note the point on the bottom of the box that the pencil first touches. Here S consists of all the points on the bottom of the box. Let the rectangular area on the right represent these points. Let A and B be the events that the pencil drops into the corresponding areas illustrated on the right. This is an example of a sample space which is not finite nor even countably infinite, i.e. which is uncountable.



Remark: If the sample space S is finite or countably infinite, then every subset of S is an event. On the other hand, if S is uncountable, as in Example 3.4, then for technical reasons (which lie beyond the scope of this text) some subsets of S cannot be events. However, in all cases the events shall form a σ-algebra $\mathcal{E}$ of subsets of S.

AXIOMS OF PROBABILITY

Let S be a sample space, let $\mathcal{E}$ be the class of events, and let P be a real-valued function defined on $\mathcal{E}$. Then P is called a *probability function*, and $P(A)$ is called the *probability* of the event A if the following axioms hold:

[$\mathbf{P}_1$] For every event A, $0 \leq P(A) \leq 1$.

[$\mathbf{P}_2$] $P(S) = 1$.

[$\mathbf{P}_3$] If A and B are mutually exclusive events, then

$$P(A \cup B) \;=\; P(A) + P(B)$$

[$\mathbf{P}_4$] If $A_1, A_2, \ldots$ is a sequence of mutually exclusive events, then

$$P(A_1 \cup A_2 \cup \cdots) \;=\; P(A_1) + P(A_2) + \cdots$$

The following remarks concerning the axioms [$\mathbf{P}_3$] and [$\mathbf{P}_4$] are in order. First of all, using [$\mathbf{P}_3$] and mathematical induction we can prove that for any mutually exclusive events $A_1, A_2, \ldots, A_n$,

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) \;=\; P(A_1) + P(A_2) + \cdots + P(A_n) \qquad (*)$$

We emphasize that [$\mathbf{P}_4$] does not follow from [$\mathbf{P}_3$] even though (*) holds for every positive integer n. However, if the sample space S is finite, then clearly the axiom [$\mathbf{P}_4$] is superfluous.

We now prove a number of theorems which follow directly from our axioms.

Theorem 3.1: If $\emptyset$ is the empty set, then $P(\emptyset) = 0$.

Proof: Let A be any set; then A and $\emptyset$ are disjoint and $A \cup \emptyset = A$. By [$\mathbf{P}_3$],

$$P(A) \;=\; P(A \cup \emptyset) \;=\; P(A) + P(\emptyset)$$

Subtracting $P(A)$ from both sides gives our result.

Theorem 3.2: If A^c is the complement of an event A, then $P(A^c) \;=\; 1 - P(A)$.

Proof: The sample space S can be decomposed into the mutually exclusive events A and A^c; that is, $S = A \cup A^c$. By [$\mathbf{P}_2$] and [$\mathbf{P}_3$] we obtain

$$1 \;=\; P(S) \;=\; P(A \cup A^c) \;=\; P(A) + P(A^c)$$

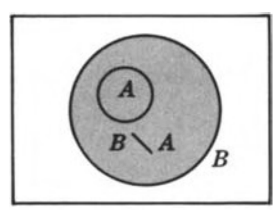
from which our result follows.

Theorem 3.3: If $A \subset B$, then $P(A) \leq P(B)$.

Proof. If $A \subset B$, then B can be decomposed into the mutually exclusive events A and $B \setminus A$ (as illustrated on the right). Thus

$$P(B) \;=\; P(A) + P(B \setminus A)$$

The result now follows from the fact that $P(B \setminus A) \geq 0$.



B is shaded.

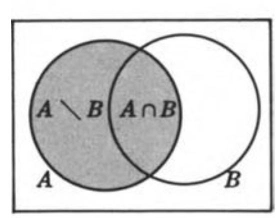
Theorem 3.4: If A and B are any two events, then

$$P(A \setminus B) \;=\; P(A) - P(A \cap B)$$

Proof. Now A can be decomposed into the mutually exclusive events $A \setminus B$ and $A \cap B$; that is, $A = (A \setminus B) \cup (A \cap B)$. Thus by [$\mathbf{P}_3$],

$$P(A) \;=\; P(A \setminus B) + P(A \cap B)$$

from which our result follows.

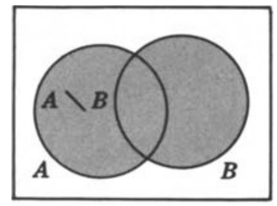


A is shaded.

Theorem 3.5: If A and B are any two events, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Proof. Note that $A \cup B$ can be decomposed into the mutually exclusive events $A \setminus B$ and B; that is, $A \cup B = (A \setminus B) \cup B$. Thus by [$\mathbf{P}_3$] and Theorem 3.4,



$A \cup B$ is shaded.

$$\begin{aligned} P(A \cup B) &= P(A \setminus B) + P(B) \\ &= P(A) - P(A \cap B) + P(B) \\ &= P(A) + P(B) - P(A \cap B) \end{aligned}$$

which is the desired result.

Applying the above theorem twice (Problem 3.23) we obtain

Corollary 3.6: For any events A, B and C,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

FINITE PROBABILITY SPACES

Let S be a finite sample space; say, $S = \{a_1, a_2, \ldots, a_n\}$. A finite probability space is obtained by assigning to each point $a_i \in S$ a real number p_i, called the *probability* of a_i, satisfying the following properties:

(i) each p_i is nonnegative, $p_i \geq 0$

(ii) the sum of the p_i is one, $p_1 + p_2 + \cdots + p_n = 1$.

The *probability* $P(A)$ of any event A, is then defined to be the sum of the probabilities of the points in A. For notational convenience we write $P(a_i)$ for $P(\{a_i\})$.

Example 3.5: Let three coins be tossed and the number of heads observed; then the sample space is $S = \{0, 1, 2, 3\}$. We obtain a probability space by the following assignment

$$P(0) = \tfrac{1}{8}, \quad P(1) = \tfrac{3}{8}, \quad P(2) = \tfrac{3}{8} \quad \text{and} \quad P(3) = \tfrac{1}{8}$$

since each probability is nonnegative and the sum of the probabilities is 1. Let A be the event that at least one head appears and let B be the event that all heads or all tails appear:

$$A = \{1, 2, 3\} \quad \text{and} \quad B = \{0, 3\}$$

Then, by definition,

$$P(A) = P(1) + P(2) + P(3) = \tfrac{3}{8} + \tfrac{3}{8} + \tfrac{1}{8} = \tfrac{7}{8}$$

and

$$P(B) = P(0) + P(3) = \tfrac{1}{8} + \tfrac{1}{8} = \tfrac{1}{4}$$

Example 3.6: Three horses A, B and C are in a race; A is twice as likely to win as B and B is twice as likely to win as C. What are their respective probabilities of winning, i.e. $P(A)$, $P(B)$ and $P(C)$?

Let $P(C) = p$; since B is twice as likely to win as C, $P(B) = 2p$; and since A is twice as likely to win as B, $P(A) = 2P(B) = 2(2p) = 4p$. Now the sum of the probabilities must be 1; hence

$$p + 2p + 4p = 1 \quad \text{or} \quad 7p = 1 \quad \text{or} \quad p = \tfrac{1}{7}$$

Accordingly,

$$P(A) = 4p = \tfrac{4}{7}, \quad P(B) = 2p = \tfrac{2}{7}, \quad P(C) = p = \tfrac{1}{7}$$

Question: What is the probability that B or C wins, i.e. $P(\{B, C\})$? By definition

$$P(\{B, C\}) = P(B) + P(C) = \tfrac{2}{7} + \tfrac{1}{7} = \tfrac{3}{7}$$

FINITE EQUIPROBABLE SPACES

Frequently, the physical characteristics of an experiment suggest that the various outcomes of the sample space be assigned equal probabilities. Such a finite probability space S, where each sample point has the same probability, will be called an *equiprobable* or *uniform space*. In particular, if S contains n points then the probability of each point is $1/n$. Furthermore, if an event A contains r points then its probability is $r \cdot \frac{1}{n} = \frac{r}{n}$. In other words,

$$P(A) = \frac{\text{number of elements in } A}{\text{number of elements in } S}$$

or

$$P(A) = \frac{\text{number of ways that the event } A \text{ can occur}}{\text{number of ways that the sample space } S \text{ can occur}}$$

We emphasize that the above formula for $P(A)$ can only be used with respect to an equiprobable space, and cannot be used in general.

The expression "at random" will be used only with respect to an equiprobable space; formally, the statement "choose a point at random from a set S" shall mean that S is an equiprobable space, i.e. that each sample point in S has the same probability.

Example 3.7: Let a card be selected at random from an ordinary deck of 52 cards. Let

$$A = \{\text{the card is a spade}\}$$

and $$B = \{\text{the card is a face card, i.e. a jack, queen or king}\}$$

We compute $P(A)$, $P(B)$ and $P(A \cap B)$. Since we have an equiprobable space,

$$P(A) = \frac{\text{number of spades}}{\text{number of cards}} = \frac{13}{52} = \frac{1}{4} \qquad P(B) = \frac{\text{number of face cards}}{\text{number of cards}} = \frac{12}{52} = \frac{3}{13}$$

$$P(A \cap B) = \frac{\text{number of spade face cards}}{\text{number of cards}} = \frac{3}{52}$$

Example 3.8: Let 2 items be chosen at random from a lot containing 12 items of which 4 are defective. Let

$$A = \{\text{both items are defective}\} \quad \text{and} \quad B = \{\text{both items are non-defective}\}$$

Find $P(A)$ and $P(B)$. Now

S can occur in $\binom{12}{2}$ = 66 ways, the number of ways that 2 items can be chosen from 12 items;

A can occur in $\binom{4}{2}$ = 6 ways, the number of ways that 2 defective items can be chosen from 4 defective items;

B can occur in $\binom{8}{2}$ = 28 ways, the number of ways that 2 non-defective items can be chosen from 8 non-defective items.

Accordingly, $P(A) = \frac{6}{66} = \frac{1}{11}$ and $P(B) = \frac{28}{66} = \frac{14}{33}$.

Question: What is the probability that at least one item is defective? Now

$$C = \{\text{at least one item is defective}\}$$

is the complement of B; that is, $C = B^c$. Thus by Theorem 3.2,

$$P(C) = P(B^c) = 1 - P(B) = 1 - \tfrac{14}{33} = \tfrac{19}{33}$$

The *odds* that an event with probability p occurs is defined to be the ratio $p : (1-p)$. Thus the odds that at least one item is defective is $\frac{19}{33} : \frac{14}{33}$ or $19 : 14$ which is read "19 to 14".

Example 3.9: (Classical Birthday Problem.) We seek the probability p that n people have distinct birthdays. In solving this problem, we ignore leap years and assume that a person's birthday can fall on any day with the same probability.

Since there are n people and 365 different days, there are 365^n ways in which the n people can have their birthdays. On the other hand, if the n persons are to have distinct birthdays, then the first person can be born on any of the 365 days, the second person can be born on the remaining 364 days, the third person can be born on the remaining 363 days, etc. Thus there are $365 \cdot 364 \cdot 363 \cdots (365-n+1)$ ways the n persons can have distinct birthdays. Accordingly,

$$p = \frac{365 \cdot 364 \cdot 363 \cdots (365-n+1)}{365^n} = \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdots \frac{365-n+1}{365}$$

It can be shown that for $n \geq 23$, $p < \frac{1}{2}$; in other words, amongst 23 or more people it is more likely that at least two of them have the same birthday than that they all have distinct birthdays.

INFINITE SAMPLE SPACES

Now suppose S is a countably infinite sample space; say $S = \{a_1, a_2, \ldots\}$. As in the finite case, we obtain a probability space by assigning to each $a_i \in S$ a real number p_i, called its probability, such that

$$\text{(i)}\ \ p_i \geq 0 \quad \text{and} \quad \text{(ii)}\ \ p_1 + p_2 + \cdots = \sum_{i=1}^{\infty} p_i = 1$$

The probability $P(A)$ of any event A is then the sum of the probabilities of its points.

Example 3.10: Consider the sample space $S = \{1, 2, 3, \ldots, \infty\}$ of the experiment of tossing a coin till a head appears; here n denotes the number of times the coin is tossed. A probability space is obtained by setting

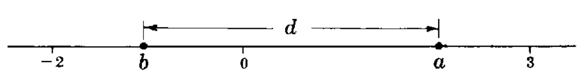
$$p(1) = \tfrac{1}{2}, \quad p(2) = \tfrac{1}{4}, \quad \ldots, \quad p(n) = 1/2^n, \quad \ldots, \quad p(\infty) = 0$$

The only uncountable sample spaces S which we will consider here are those with some finite geometrical measurement $m(S)$ such as length, area or volume, and in which a point is selected at random. The probability of an event A, i.e. that the selected point belongs to A, is then the ratio of $m(A)$ to $m(S)$; that is,

$$P(A) = \frac{\text{length of } A}{\text{length of } S} \quad \text{or} \quad P(A) = \frac{\text{area of } A}{\text{area of } S} \quad \text{or} \quad P(A) = \frac{\text{volume of } A}{\text{volume of } S}$$

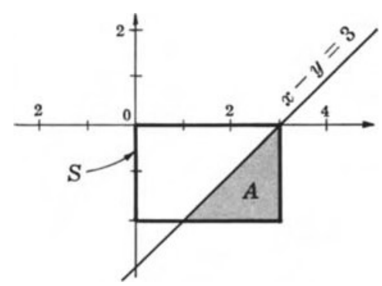
Such a probability space is said to be *uniform.*

Example 3.11: On the real line $\mathbf{R}$, points a and b are selected at random such that $-2 \leq b \leq 0$ and $0 \leq a \leq 3$, as shown below. Find the probability p that the distance d between a and b is greater than 3.



The sample space S consists of the ordered pairs (a, b) and so forms the rectangular region shown in the adjacent diagram. On the other hand, the set A of points (a, b) for which $d = a - b > 3$ consists of those points of S which lie below the line $x - y = 3$, and hence forms the shaded area in the diagram. Thus

$$p = P(A) = \frac{\text{area of } A}{\text{area of } S} = \frac{2}{6} = \frac{1}{3}$$



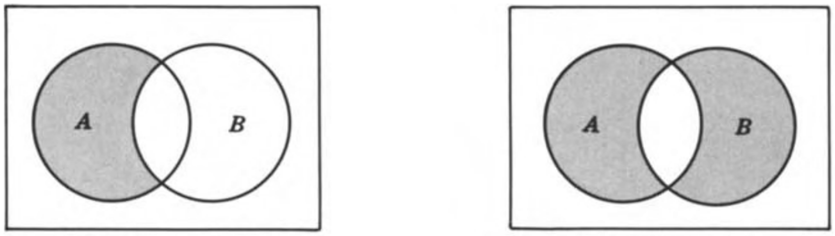
Remark: A finite or countably infinite probability space is said to be *discrete*, and an uncountable space is said to be *nondiscrete*.

Solved Problems

SAMPLE SPACES AND EVENTS

3.1. Let A and B be events. Find an expression and exhibit the Venn diagram for the event that: (i) A but not B occurs, i.e. only A occurs; (ii) either A or B, but not both, occurs, i.e. exactly one of the two events occurs.

(i) Since A but not B occurs, shade the area of A outside of B as in Figure (a) below. Note that B^c, the complement of B, occurs since B does not occur; hence A and B^c occurs. In other words, the event is $A \cap B^c$.



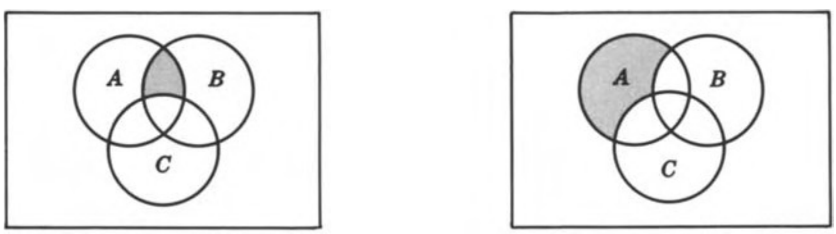
A but not B occurs. (a)

Either A or B, but not both, occurs. (b)

(ii) Since A or B but not both occurs, shade the area of A and B except where they intersect as in Figure (b) above. The event is equivalent to A but not B occurs or B but not A occurs. Now, as in (i), A but not B is the event $A \cap B^c$, and B but not A is the event $B \cap A^c$. Thus the given event is $(A \cap B^c) \cup (B \cap A^c)$.

3.2. Let A, B and C be events. Find an expression and exhibit the Venn diagram for the event that (i) A and B but not C occurs, (ii) only A occurs.

(i) Since A and B but not C occurs, shade the intersection of A and B which lies outside of C, as in Figure (a) below. The event is $A \cap B \cap C^c$.



A and B but not C occurs. (a)

Only A occurs. (b)

(ii) Since only A is to occur, shade the area of A which lies outside of B and of C, as in Figure (b) above. The event is $A \cap B^c \cap C^c$.

3.3. Let a coin and a die be tossed; let the sample space S consist of the twelve elements:

$$S = \{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\}$$

(i) Express explicitly the following events: A = {heads and an even number appear}, B = {a prime number appears}, C = {tails and an odd number appear}.

(ii) Express explicitly the event that: (a) A or B occurs, (b) B and C occurs, (c) only B occurs.

(iii) Which of the events A, B and C are mutually exclusive?

(i) To obtain A, choose those elements of S consisting of an H and an even number: $A = \{H2, H4, H6\}$.

To obtain B, choose those points in S consisting of a prime number: $B = \{H2, H3, H5, T2, T3, T5\}$.

To obtain C, choose those points in S consisting of a T and an odd number: $C = \{T1, T3, T5\}$.

(ii) (a) A or $B = A \cup B = \{H2, H4, H6, H3, H5, T2, T3, T5\}$

(b) B and $C = B \cap C = \{T3, T5\}$

(c) Choose those elements of B which do not lie in A or C: $B \cap A^c \cap C^c = \{H3, H5, T2\}$.

(iii) A and C are mutually exclusive since $A \cap C = \emptyset$.

FINITE PROBABILITY SPACES

3.4. Suppose a sample space S consists of 4 elements: $S = \{a_1, a_2, a_3, a_4\}$. Which function defines a probability space on S?

(i) $P(a_1) = \frac{1}{2}$, $P(a_2) = \frac{1}{3}$, $P(a_3) = \frac{1}{4}$, $P(a_4) = \frac{1}{5}$.

(ii) $P(a_1) = \frac{1}{2}$, $P(a_2) = \frac{1}{4}$, $P(a_3) = -\frac{1}{4}$, $P(a_4) = \frac{1}{2}$.

(iii) $P(a_1) = \frac{1}{2}$, $P(a_2) = \frac{1}{4}$, $P(a_3) = \frac{1}{8}$, $P(a_4) = \frac{1}{8}$.

(iv) $P(a_1) = \frac{1}{2}$, $P(a_2) = \frac{1}{4}$, $P(a_3) = \frac{1}{4}$, $P(a_4) = 0$.

(i) Since the sum of the values on the sample points is greater than one, $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{77}{60}$, the function does not define a probability space on S.

(ii) Since $P(a_3) = -\frac{1}{4}$, a negative number, the function does not define a probability space on S.

(iii) Since each value is nonnegative, and the sum of the values is one, $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = 1$, the function does define a probability space on S.

(iv) The values are nonnegative and add up to one; hence the function does define a probability space on S.

3.5. Let $S = \{a_1, a_2, a_3, a_4\}$, and let P be a probability function on S.

(i) Find $P(a_1)$ if $P(a_2) = \frac{1}{3}$, $P(a_3) = \frac{1}{6}$, $P(a_4) = \frac{1}{9}$.

(ii) Find $P(a_1)$ and $P(a_2)$ if $P(a_3) = P(a_4) = \frac{1}{4}$ and $P(a_1) = 2P(a_2)$.

(iii) Find $P(a_1)$ if $P(\{a_2, a_3\}) = \frac{2}{3}$, $P(\{a_2, a_4\}) = \frac{1}{2}$ and $P(a_2) = \frac{1}{3}$.

(i) Let $P(a_1) = p$. Then for P to be a probability function, the sum of the probabilities on the sample points must be one: $p + \frac{1}{3} + \frac{1}{6} + \frac{1}{9} = 1$ or $p = \frac{7}{18}$.

(ii) Let $P(a_2) = p$, then $P(a_1) = 2p$. Hence $2p + p + \frac{1}{4} + \frac{1}{4} = 1$ or $p = \frac{1}{6}$. Thus $P(a_2) = \frac{1}{6}$ and $P(a_1) = \frac{1}{3}$.

(iii) Let $P(a_1) = p$.

$$P(a_3) = P(\{a_2, a_3\}) - P(a_2) = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

$$P(a_4) = P(\{a_2, a_4\}) - P(a_2) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

Then $p + \frac{1}{3} + \frac{1}{3} + \frac{1}{6} = 1$ or $p = \frac{1}{6}$, that is, $P(a_1) = \frac{1}{6}$.

3.6. A coin is weighted so that heads is twice as likely to appear as tails. Find $P(T)$ and $P(H)$.

Let $P(T) = p$; then $P(H) = 2p$. Now set the sum of the probabilities equal to one: $p + 2p = 1$ or $p = \frac{1}{3}$. Thus $P(T) = p = \frac{1}{3}$ and $P(H) = 2p = \frac{2}{3}$.

3.7. Two men, m_1 and m_2, and three women, w_1, w_2 and w_3, are in a chess tournament. Those of the same sex have equal probabilities of winning, but each man is twice as likely to win as any woman. (i) Find the probability that a woman wins the tournament. (ii) If m_1 and w_1 are married, find the probability that one of them wins the tournament.

Set $P(w_1) = p$; then $P(w_2) = P(w_3) = p$ and $P(m_1) = P(m_2) = 2p$. Next set the sum of the probabilities of the five sample points equal to one: $p + p + p + 2p + 2p = 1$ or $p = \frac{1}{7}$.

We seek (i) $P(\{w_1, w_2, w_3\})$ and (ii) $P(\{m_1, w_1\})$. Then by definition,

$$P(\{w_1, w_2, w_3\}) = P(w_1) + P(w_2) + P(w_3) = \tfrac{1}{7} + \tfrac{1}{7} + \tfrac{1}{7} = \tfrac{3}{7}$$

$$P(\{m_1, w_1\}) = P(m_1) + P(w_1) = \tfrac{2}{7} + \tfrac{1}{7} = \tfrac{3}{7}$$

3.8. Let a die be weighted so that the probability of a number appearing when the die is tossed is proportional to the given number (e.g. 6 has twice the probability of appearing as 3). Let $A = \{\text{even number}\}$, $B = \{\text{prime number}\}$, $C = \{\text{odd number}\}$.

(i) Describe the probability space, i.e. find the probability of each sample point.

(ii) Find $P(A)$, $P(B)$ and $P(C)$.

(iii) Find the probability that: (*a*) an even or prime number occurs; (*b*) an odd prime number occurs; (*c*) A but not B occurs.

(i) Let $P(1) = p$. Then $P(2) = 2p$, $P(3) = 3p$, $P(4) = 4p$, $P(5) = 5p$ and $P(6) = 6p$. Since the sum of the probabilities must be one, we obtain $p + 2p + 3p + 4p + 5p + 6p = 1$ or $p = 1/21$. Thus

$$P(1) = \tfrac{1}{21},\quad P(2) = \tfrac{2}{21},\quad P(3) = \tfrac{1}{7},\quad P(4) = \tfrac{4}{21},\quad P(5) = \tfrac{5}{21},\quad P(6) = \tfrac{2}{7}$$

(ii) $P(A) = P(\{2,4,6\}) = \frac{4}{7}$, $P(B) = P(\{2,3,5\}) = \frac{10}{21}$, $P(C) = P(\{1,3,5\}) = \frac{3}{7}$.

(iii) (*a*) The event that an even or prime number occurs is $A \cup B = \{2,4,6,3,5\}$, or that 1 does not occur. Thus $P(A \cup B) = 1 - P(1) = \frac{20}{21}$.

(*b*) The event that an odd prime number occurs is $B \cap C = \{3,5\}$. Thus $P(B \cap C) = P(\{3,5\}) = \frac{8}{21}$.

(*c*) The event that A but not B occurs is $A \cap B^c = \{4,6\}$. Hence $P(A \cap B^c) = P(\{4,6\}) = \frac{10}{21}$.

FINITE EQUIPROBABLE SPACES

3.9. Determine the probability p of each event:

(i) an even number appears in the toss of a fair die;

(ii) a king appears in drawing a single card from an ordinary deck of 52 cards;

(iii) at least one tail appears in the toss of three fair coins;

(iv) a white marble appears in drawing a single marble from an urn containing 4 white, 3 red and 5 blue marbles.

(i) The event can occur in three ways (a 2, 4 or 6) out of 6 equally likely cases; hence $p = \frac{3}{6} = \frac{1}{2}$.

(ii) There are 4 kings among the 52 cards; hence $p = \frac{4}{52} = \frac{1}{13}$.

(iii) If we consider the coins distinguished, then there are 8 equally likely cases: HHH, HHT, HTH, HTT, THH, THT, TTH, TTT. Only the first case is not favorable to the given event; hence $p = \frac{7}{8}$.

(iv) There are $4 + 3 + 5 = 12$ marbles, of which 4 are white; hence $p = \frac{4}{12} = \frac{1}{3}$.

3.10. Two cards are drawn at random from an ordinary deck of 52 cards. Find the probability p that (i) both are spades, (ii) one is a spade and one is a heart.

There are $\binom{52}{2} = 1326$ ways to draw 2 cards from 52 cards.

(i) There are $\binom{13}{2} = 78$ ways to draw 2 spades from 13 spades; hence

$$p = \frac{\text{number of ways 2 spades can be drawn}}{\text{number of ways 2 cards can be drawn}} = \frac{78}{1326} = \frac{1}{17}$$

(ii) Since there are 13 spades and 13 hearts, there are $13 \cdot 13 = 169$ ways to draw a spade and a heart; hence $p = \frac{169}{1326} = \frac{13}{102}$.

3.11. Three light bulbs are chosen at random from 15 bulbs of which 5 are defective. Find the probability p that (i) none is defective, (ii) exactly one is defective, (iii) at least one is defective.

There are $\binom{15}{3} = 455$ ways to choose 3 bulbs from the 15 bulbs.

(i) Since there are $15 - 5 = 10$ nondefective bulbs, there are $\binom{10}{3} = 120$ ways to choose 3 nondefective bulbs. Thus $p = \frac{120}{455} = \frac{24}{91}$.

(ii) There are 5 defective bulbs and $\binom{10}{2} = 45$ different pairs of nondefective bulbs; hence there are $5 \cdot 45 = 225$ ways to choose 3 bulbs of which one is defective. Thus $p = \frac{225}{455} = \frac{45}{91}$.

(iii) The event that at least one is defective is the complement of the event that none are defective which has, by (i), probability $\frac{24}{91}$. Hence $p = 1 - \frac{24}{91} = \frac{67}{91}$.

3.12. Two cards are selected at random from 10 cards numbered 1 to 10. Find the probability p that the sum is odd if (i) the two cards are drawn together, (ii) the two cards are drawn one after the other without replacement, (iii) the two cards are drawn one after the other with replacement.

(i) There are $\binom{10}{2} = 45$ ways to select 2 cards out of 10. The sum is odd if one number is odd and the other is even. There are 5 even numbers and 5 odd numbers; hence there are $5 \cdot 5 = 25$ ways of choosing an even and an odd number. Thus $p = \frac{25}{45} = \frac{5}{9}$.

(ii) There are $10 \cdot 9 = 90$ ways to draw two cards one after the other without replacement. There are $5 \cdot 5 = 25$ ways to draw an even number and then an odd number, and $5 \cdot 5 = 25$ ways to draw an odd number and then an even number; hence $p = \frac{25+25}{90} = \frac{50}{90} = \frac{5}{9}$.

(iii) There are $10 \cdot 10 = 100$ ways to draw two cards one after the other with replacement. As in (ii), there are $5 \cdot 5 = 25$ ways to draw an even number and then an odd number, and $5 \cdot 5 = 25$ ways to draw an odd number and then an even number; hence $p = \frac{25+25}{100} = \frac{50}{100} = \frac{1}{2}$.

3.13. Six married couples are standing in a room.

(i) If 2 people are chosen at random, find the probability p that (*a*) they are married, (*b*) one is male and one is female.

(ii) If 4 people are chosen at random, find the probability p that (*a*) 2 married couples are chosen, (*b*) no married couple is among the 4, (*c*) exactly one married couple is among the 4.

(iii) If the 12 people are divided into six pairs, find the probability p that (*a*) each pair is married, (*b*) each pair contains a male and a female.

(i) There are $\binom{12}{2} = 66$ ways to choose 2 people from the 12 people.

(a) There are 6 married couples; hence $p = \frac{6}{66} = \frac{1}{11}$.

(b) There are 6 ways to choose a male and 6 ways to choose a female; hence $p = \frac{6 \cdot 6}{66} = \frac{6}{11}$.

(ii) There are $\binom{12}{4} = 495$ ways to choose 4 people from the 12 people.

(a) There are $\binom{6}{2} = 15$ ways to choose 2 couples from the 6 couples; hence $p = \frac{15}{495} = \frac{1}{33}$.

(b) The 4 persons come from 4 different couples. There are $\binom{6}{4} = 15$ ways to choose 4 couples from the 6 couples, and there are 2 ways to choose one person from each couple. Hence $p = \frac{2 \cdot 2 \cdot 2 \cdot 2 \cdot 15}{495} = \frac{16}{33}$.

(c) This event is mutually disjoint from the preceding two events (which are also mutually disjoint) and at least one of these events must occur. Hence $p + \frac{1}{33} + \frac{16}{33} = 1$ or $p = \frac{16}{33}$.

(iii) There are $\frac{12!}{2!\,2!\,2!\,2!\,2!\,2!} = \frac{12!}{2^6}$ ways to partition the 12 people into 6 ordered cells with 2 people in each.

(a) The 6 couples can be placed into the 6 ordered cells in 6! ways. Hence $p = \frac{6!}{12!/2^6} = \frac{1}{10{,}395}$.

(b) The six men can be placed one each into the 6 cells in 6! ways, and the 6 women can be placed one each into the 6 cells in 6! ways. Hence $p = \frac{6!\,6!}{12!/2^6} = \frac{16}{231}$.

3.14. A class contains 10 men and 20 women of which half the men and half the women have brown eyes. Find the probability p that a person chosen at random is a man or has brown eyes.

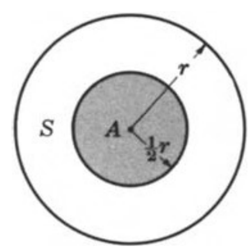
Let $A = \{\text{person is a man}\}$ and $B = \{\text{person has brown eyes}\}$. We seek $P(A \cup B)$.

Then $P(A) = \frac{10}{30} = \frac{1}{3}$, $P(B) = \frac{15}{30} = \frac{1}{2}$, $P(A \cap B) = \frac{5}{30} = \frac{1}{6}$. Thus by Theorem 3.5,

$$p = P(A \cup B) = P(A) + P(B) - P(A \cap B) = \tfrac{1}{3} + \tfrac{1}{2} - \tfrac{1}{6} = \tfrac{2}{3}$$

UNCOUNTABLE UNIFORM SPACES

3.15. A point is selected at random inside a circle. Find the probability p that the point is closer to the center of the circle than to its circumference.



Let S denote the set of points inside the circle with radius r, and let A denote the set of points inside the concentric circle of radius $\frac{1}{2}r$. (Thus A consists precisely of those points of S which are closer to its center than to its circumference.) Accordingly,

$$p = P(A) = \frac{\text{area of } A}{\text{area of } S} = \frac{\pi(\frac{1}{2}r)^2}{\pi r^2} = \frac{1}{4}$$

3.16. Consider the Cartesian plane $\mathbf{R}^2$, and let X denote the subset of points for which both coordinates are integers. A coin of diameter $\frac{1}{2}$ is tossed randomly onto the plane. Find the probability p that the coin covers a point of X.

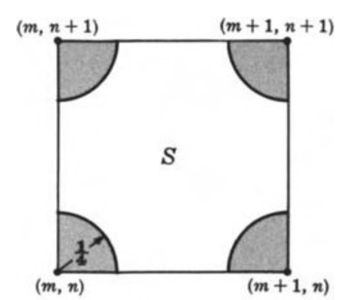
Let S denote the set of points inside a square with corners

$$(m, n), \quad (m, n+1), \quad (m+1, n), \quad (m+1, n+1) \in X$$

Let A denote the set of points in S with distance less than $\frac{1}{4}$ from any corner point. (Observe that the area of A is equal to the area inside a circle of radius $\frac{1}{4}$.) Thus a coin whose center falls in S will cover a point of X if and only if its center falls in a point of A. Accordingly,

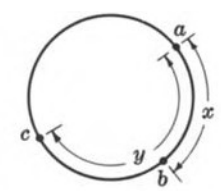
$$p = P(A) = \frac{\text{area of } A}{\text{area of } S} = \frac{\pi(\frac{1}{4})^2}{1} = \frac{\pi}{16} \approx .2$$

Note. We cannot take S to be all of $\mathbf{R}^2$ because the latter has infinite area.



A is shaded.

3.17. Three points a, b and c are selected at random from the circumference of a circle. Find the probability p that the points lie on a semicircle.



Suppose the length of the circumference is $2s$. Let x denote the clockwise arc length from a to b, and let y denote the clockwise arc length from a to c. Thus

$$0 < x < 2s \quad \text{and} \quad 0 < y < 2s \qquad (*)$$

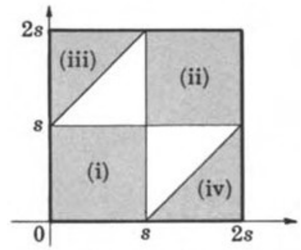
Let S denote the set of points in $\mathbf{R}^2$ for which condition (*) holds. Let A denote the subset of S for which any of the following conditions holds:

(i) $x, y < s$ (iii) $x < s$ and $y - x > s$

(ii) $x, y > s$ (iv) $y < s$ and $x - y > s$

Then A consists of those points for which a, b and c lie on a semicircle. Thus

$$p = \frac{\text{area of } A}{\text{area of } S} = \frac{3s^2}{4s^2} = \frac{3}{4}$$



A is shaded.

MISCELLANEOUS PROBLEMS

3.18. Let A and B be events with $P(A) = \frac{3}{8}$, $P(B) = \frac{1}{2}$ and $P(A \cap B) = \frac{1}{4}$. Find (i) $P(A \cup B)$, (ii) $P(A^c)$ and $P(B^c)$, (iii) $P(A^c \cap B^c)$, (iv) $P(A^c \cup B^c)$, (v) $P(A \cap B^c)$, (vi) $P(B \cap A^c)$.

(i) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{3}{8} + \frac{1}{2} - \frac{1}{4} = \frac{5}{8}$

(ii) $P(A^c) = 1 - P(A) = 1 - \frac{3}{8} = \frac{5}{8}$ and $P(B^c) = 1 - P(B) = 1 - \frac{1}{2} = \frac{1}{2}$

(iii) Using De Morgan's Law, $(A \cup B)^c = A^c \cap B^c$, we have

$$P(A^c \cap B^c) = P((A \cup B)^c) = 1 - P(A \cup B) = 1 - \frac{5}{8} = \frac{3}{8}$$

(iv) Using De Morgan's Law, $(A \cap B)^c = A^c \cup B^c$, we have

$$P(A^c \cup B^c) = P((A \cap B)^c) = 1 - P(A \cap B) = 1 - \frac{1}{4} = \frac{3}{4}$$

Equivalently,

$$P(A^c \cup B^c) = P(A^c) + P(B^c) - P(A^c \cap B^c) = \frac{5}{8} + \frac{1}{2} - \frac{3}{8} = \frac{3}{4}$$

(v) $P(A \cap B^c) = P(A \setminus B) = P(A) - P(A \cap B) = \frac{3}{8} - \frac{1}{4} = \frac{1}{8}$

(vi) $P(B \cap A^c) = P(B) - P(A \cap B) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

3.19. Let A and B be events with $P(A \cup B) = \frac{3}{4}$, $P(A^c) = \frac{2}{3}$ and $P(A \cap B) = \frac{1}{4}$. Find (i) $P(A)$, (ii) $P(B)$, (iii) $P(A \cap B^c)$.

(i) $P(A) = 1 - P(A^c) = 1 - \frac{2}{3} = \frac{1}{3}$

(ii) Substitute in $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ to obtain $\frac{3}{4} = \frac{1}{3} + P(B) - \frac{1}{4}$ or $P(B) = \frac{2}{3}$.

(iii) $P(A \cap B^c) = P(A) - P(A \cap B) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

3.20. Find the probability p of an event if the odds that it will occur are $a : b$, that is, "a to b".

The odds that an event with probability p occurs is the ratio $p : (1 - p)$. Hence

$$\frac{p}{1-p} = \frac{a}{b} \quad \text{or} \quad bp = a - ap \quad \text{or} \quad ap + bp = a \quad \text{or} \quad p = \frac{a}{a+b}$$

3.21. Find the probability p of an event if the odds that it will occur are "3 to 2".

$\frac{p}{1-p} = \frac{3}{2}$ from which $p = \frac{3}{5}$. We can also use the formula of the preceding problem to obtain the answer directly: $p = \frac{a}{a+b} = \frac{3}{3+2} = \frac{3}{5}$.

3.22. A die is tossed 100 times. The following table lists the six numbers and frequency with which each number appeared:

Number	1	2	3	4	5	6
Frequency	14	17	20	18	15	16

Find the relative frequency f of the event (i) a 3 appears, (ii) a 5 appears, (iii) an even number appears, (iv) a prime appears.

The relative frequency $f = \dfrac{\text{number of successes}}{\text{total number of trials}}$.

(i) $f = \frac{20}{100} = .20$ (ii) $f = \frac{15}{100} = .15$ (iii) $f = \frac{17+18+16}{100} = .51$ (iv) $f = \frac{17+20+15}{100} = .52$

3.23. Prove Corollary 3.6: For any events A, B and C,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

Let $D = B \cup C$. Then $A \cap D = A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and

$$P(A \cap D) = P(A \cap B) + P(A \cap C) - P(A \cap B \cap A \cap C) = P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)$$

Thus

$$\begin{aligned} P(A \cup B \cup C) &= P(A \cup D) = P(A) + P(D) - P(A \cap D) \\ &= P(A) + P(B) + P(C) - P(B \cap C) - [P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)] \\ &= P(A) + P(B) + P(C) - P(B \cap C) - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C) \end{aligned}$$

3.24. Let $S = \{a_1, a_2, \ldots, a_s\}$ and $T = \{b_1, b_2, \ldots, b_t\}$ be finite probability spaces. Let the number $p_{ij} = P(a_i)\,P(b_j)$ be assigned to the ordered pair (a_i, b_j) in the product set $S \times T = \{(s, t) : s \in S,\ t \in T\}$. Show that the p_{ij} define a probability space on $S \times T$, i.e. that the p_{ij} are nonnegative and add up to one. (This is called the *product probability space*. We emphasize that this is not the only probability function that can be defined on the product set $S \times T$.)

Since $P(a_i), P(b_j) \geq 0$, for each i and each j, $p_{ij} = P(a_i)\,P(b_j) \geq 0$. Furthermore,

$$\begin{aligned} & p_{11} + p_{12} + \cdots + p_{1t} + p_{21} + p_{22} + \cdots + p_{2t} + \cdots + p_{s1} + p_{s2} + \cdots + p_{st} \\ &= P(a_1)\,P(b_1) + \cdots + P(a_1)\,P(b_t) + \cdots + P(a_s)\,P(b_1) + \cdots + P(a_s)\,P(b_t) \\ &= P(a_1)[P(b_1) + \cdots + P(b_t)] + \cdots + P(a_s)[P(b_1) + \cdots + P(b_t)] \\ &= P(a_1) \cdot 1 + \cdots + P(a_s) \cdot 1 \\ &= P(a_1) + \cdots + P(a_s) \\ &= 1 \end{aligned}$$

Supplementary Problems

SAMPLE SPACES AND EVENTS

3.25. Let A and B be events. Find an expression and exhibit the Venn diagram for the event that (i) A or not B occurs, (ii) neither A nor B occurs.

3.26. Let A, B and C be events. Find an expression and exhibit the Venn diagram for the event that (i) exactly one of the three events occurs, (ii) at least two of the events occurs, (iii) none of the events occurs, (iv) A or B, but not C, occurs.

3.27. Let a penny, a dime and a die be tossed.

(i) Describe a suitable sample space S.

(ii) Express explicitly the following events: A = {two heads and an even number appear}, B = {a 2 appears}, C = {exactly one head and a prime number appear}.

(iii) Express explicitly the event that (*a*) A and B occur, (*b*) only B occurs, (*c*) B or C occurs.

FINITE PROBABILITY SPACES

3.28. Which function defines a probability space on $S = \{a_1, a_2, a_3\}$?

(i) $P(a_1) = \frac{1}{4},\ P(a_2) = \frac{1}{3},\ P(a_3) = \frac{1}{2}$ (iii) $P(a_1) = \frac{1}{6},\ P(a_2) = \frac{1}{3},\ P(a_3) = \frac{1}{2}$

(ii) $P(a_1) = \frac{2}{3},\ P(a_2) = -\frac{1}{3},\ P(a_3) = \frac{2}{3}$ (iv) $P(a_1) = 0,\ P(a_2) = \frac{1}{3},\ P(a_3) = \frac{2}{3}$

3.29. Let P be a probability function on $S = \{a_1, a_2, a_3\}$. Find $P(a_1)$ if (i) $P(a_2) = \frac{1}{3}$ and $P(a_3) = \frac{1}{4}$, (ii) $P(a_1) = 2\,P(a_2)$ and $P(a_3) = \frac{1}{4}$, (iii) $P(\{a_2, a_3\}) = 2\,P(a_1)$, (iv) $P(a_3) = 2\,P(a_2)$ and $P(a_2) = 3\,P(a_1)$.

3.30. A coin is weighted so that heads is three times as likely to appear as tails. Find P(H) and P(T).

3.31. Three students A, B and C are in a swimming race. A and B have the same probability of winning and each is twice as likely to win as C. Find the probability that B or C wins.

3.32. A die is weighted so that the even numbers have the same chance of appearing, the odd numbers have the same chance of appearing, and each even number is twice as likely to appear as any odd number. Find the probability that (i) an even number appears, (ii) a prime number appears, (iii) an odd number appears, (iv) an odd prime number appears.

3.33. Find the probability of an event if the odds that it will occur are (i) 2 to 1, (ii) 5 to 11.

3.34. In a swimming race, the odds that A will win are 2 to 3 and the odds that B will win are 1 to 4. Find the probability p and the odds that A or B wins the race.

FINITE EQUIPROBABLE SPACES

3.35. A class contains 5 freshmen, 4 sophomores, 8 juniors and 3 seniors. A student is chosen at random to represent the class. Find the probability that the student is (i) a sophomore, (ii) a senior, (iii) a junior or senior.

3.36. One card is selected at random from 50 cards numbered 1 to 50. Find the probability that the number on the card is (i) divisible by 5, (ii) prime, (iii) ends in the digit 2.

3.37. Of 10 girls in a class, 3 have blue eyes. If two of the girls are chosen at random, what is the probability that (i) both have blue eyes, (ii) neither has blue eyes, (iii) at least one has blue eyes?

3.38. Three bolts and three nuts are put in a box. If two parts are chosen at random, find the probability that one is a bolt and one a nut.

3.39. Ten students, $A, B, \ldots$, are in a class. If a committee of 3 is chosen at random from the class, find the probability that (i) A belongs to the committee, (ii) B belongs to the committee, (iii) A and B belong to the committee, (iv) A or B belongs to the committee.

3.40. A class consists of 6 girls and 10 boys. If a committee of 3 is chosen at random from the class, find the probability that (i) 3 boys are selected, (ii) exactly 2 boys are selected, (iii) at least one boy is selected, (iv) exactly 2 girls are selected.

3.41. A pair of fair dice is tossed. Find the probability that the maximum of the two numbers is greater than 4.

3.42. Of 120 students, 60 are studying French, 50 are studying Spanish, and 20 are studying French and Spanish. If a student is chosen at random, find the probability that the student (i) is studying French or Spanish, (ii) is studying neither French nor Spanish.

3.43. Three boys and 3 girls sit in a row. Find the probability that (i) the 3 girls sit together, (ii) the boys and girls sit in alternate seats.

NONCOUNTABLE UNIFORM SPACES

3.44. A point is selected at random inside an equilateral triangle whose side length is 3. Find the probability that its distance to any corner is greater than 1.

3.45. A coin of diameter $\frac{1}{2}$ is tossed randomly onto the Cartesian plane $\mathbf{R}^2$. Find the probability that the coin does not intersect any line whose equation is of the form (a) $x = k$, (b) $x + y = k$, (c) $x = k$ or $y = k$. (Here k is an integer.)

3.46. A point X is selected at random from a line segment AB with midpoint O. Find the probability that the line segments AX, XB and AO can form a triangle.

MISCELLANEOUS PROBLEMS

3.47. Let A and B be events with $P(A \cup B) = \frac{7}{8}$, $P(A \cap B) = \frac{1}{4}$ and $P(A^c) = \frac{5}{8}$. Find $P(A)$, $P(B)$ and $P(A \cap B^c)$.

3.48. Let A and B be events with $P(A) = \frac{1}{2}$, $P(A \cup B) = \frac{3}{4}$ and $P(B^c) = \frac{5}{8}$. Find $P(A \cap B)$, $P(A^c \cap B^c)$, $P(A^c \cup B^c)$ and $P(B \cap A^c)$.

3.49. A die is tossed 50 times. The following table gives the six numbers and their frequency of occurrence:

Number	1	2	3	4	5	6
Frequency	7	9	8	7	9	10

Find the relative frequency of the event (i) a 4 appears, (ii) an odd number appears, (iii) a prime number appears.

3.50. Prove: For any events $A_1, A_2, \ldots, A_n$,

$$P(A_1 \cup \cdots \cup A_n) = \sum_i P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - \cdots \pm P(A_1 \cap \cdots \cap A_n)$$

(*Remark*: This result generalizes Theorem 3.5 and Corollary 3.6.)

Answers to Supplementary Problems

3.25. (i) $A \cup B^c$, (ii) $(A \cup B)^c$

3.26. (i) $(A \cap B^c \cap C^c) \cup (B \cap A^c \cap C^c) \cup (C \cap A^c \cap B^c)$ (iii) $(A \cup B \cup C)^c$

(ii) $(A \cap B) \cup (A \cap C) \cup (B \cap C)$ (iv) $(A \cup B) \cap C^c$

3.27. (i) S = {HH1, HH2, HH3, HH4, HH5, HH6, HT1, HT2, HT3, HT4, HT5, HT6, TH1, TH2, TH3, TH4, TH5, TH6, TT1, TT2, TT3, TT4, TT5, TT6}

(ii) A = {HH2, HH4, HH6}, B = {HH2, HT2, TH2, TT2}, C = {HT2, TH2, HT3, TH3, HT5, TH5}

(iii) (a) $A \cap B$ = {HH2}

(b) $B \setminus (A \cup C)$ = {TT2}

(c) $B \cup C$ = {HH2, HT2, TH2, TT2, HT3, TH3, HT5, TH5}

3.28. (i) no, (ii) no, (iii) yes, (iv) yes

3.29. (i) $\frac{5}{12}$, (ii) $\frac{1}{2}$, (iii) $\frac{1}{3}$, (iv) $\frac{1}{10}$

3.30. $P(H) = \frac{3}{4}$, $P(T) = \frac{1}{4}$

3.31. $\frac{3}{5}$

3.32. (i) $\frac{2}{3}$, (ii) $\frac{4}{9}$, (iii) $\frac{1}{3}$, (iv) $\frac{2}{9}$

3.33. (i) $\frac{2}{3}$, (ii) $\frac{5}{16}$

3.34. $p = \frac{3}{5}$; the odds are 3 to 2.

3.35. (i) $\frac{1}{5}$, (ii) $\frac{3}{20}$, (iii) $\frac{11}{20}$

3.36. (i) $\frac{1}{5}$, (ii) $\frac{3}{10}$, (iii) $\frac{1}{10}$

3.37. (i) $\frac{1}{15}$, (ii) $\frac{7}{15}$, (iii) $\frac{8}{15}$

3.38. $\frac{3}{5}$

3.39. (i) $\frac{3}{10}$, (ii) $\frac{3}{10}$, (iii) $\frac{1}{15}$, (iv) $\frac{8}{15}$

3.40. (i) $\frac{3}{14}$, (ii) $\frac{27}{56}$, (iii) $\frac{27}{28}$, (iv) $\frac{15}{56}$

3.41. $\frac{5}{9}$

3.42. (i) $\frac{3}{4}$, (ii) $\frac{1}{4}$

3.43. (i) $\frac{1}{5}$, (ii) $\frac{1}{10}$

3.44. $1 - 2\pi/(9\sqrt{3})$

3.45. (i) $\frac{1}{2}$, (ii) $1 - \frac{1}{2}\sqrt{2}$, (iii) $\frac{1}{4}$

3.46. $\frac{1}{2}$

3.47. $P(A) = \frac{5}{8}$, $P(B) = \frac{3}{4}$, $P(A \cap B^c) = \frac{1}{8}$

3.48. $P(A \cap B) = \frac{1}{8}$, $P(A^c \cap B^c) = \frac{1}{4}$, $P(A^c \cup B^c) = \frac{7}{8}$, $P(B \cap A^c) = \frac{1}{4}$

3.49. (i) $\frac{7}{50}$, (ii) $\frac{24}{50}$, (iii) $\frac{26}{50}$

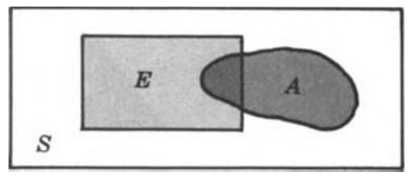
Chapter 4

Conditional Probability and Independence

CONDITIONAL PROBABILITY

Let E be an arbitrary event in a sample space S with $P(E) > 0$. The probability that an event A occurs once E has occurred or, in other words, the *conditional probability* of A given E, written $P(A \mid E)$, is defined as follows:

$$P(A \mid E) = \frac{P(A \cap E)}{P(E)}$$



As seen in the adjoining Venn diagram, $P(A \mid E)$ in a certain sense measures the relative probability of A with respect to the reduced space E.

In particular, if S is a finite equiprobable space and $|A|$ denotes the number of elements in an event A, then

$$P(A \cap E) = \frac{|A \cap E|}{|S|}, \quad P(E) = \frac{|E|}{|S|} \quad \text{and so} \quad P(A \mid E) = \frac{P(A \cap E)}{P(E)} = \frac{|A \cap E|}{|E|}$$

That is,

Theorem 4.1: Let S be a finite equiprobable space with events A and E. Then

$$P(A \mid E) = \frac{\text{number of elements in } A \cap E}{\text{number of elements in } E}$$

or

$$P(A \mid E) = \frac{\text{number of ways } A \text{ and } E \text{ can occur}}{\text{number of ways } E \text{ can occur}}$$

Example 4.1: Let a pair of fair dice be tossed. If the sum is 6, find the probability that one of the dice is a 2. In other words, if

$$E = \{\text{sum is } 6\} = \{(1,5), (2,4), (3,3), (4,2), (5,1)\}$$

and

$$A = \{\text{a 2 appears on at least one die}\}$$

find $P(A \mid E)$.

Now E consists of five elements and two of them, $(2,4)$ and $(4,2)$, belong to A: $A \cap E = \{(2,4), (4,2)\}$. Then $P(A \mid E) = \frac{2}{5}$.

On the other hand, since A consists of eleven elements,

$$A = \{(2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (1,2), (3,2), (4,2), (5,2), (6,2)\}$$

and S consists of 36 elements, $P(A) = \frac{11}{36}$.

Example 4.2: A couple has two children. Find the probability p that both children are boys if (i) we are given that the younger child is a boy, (ii) we are given that (at least) one of the children is a boy.

The sample space for the sex of two children is $S = \{bb, bg, gb, gg\}$ with probability $\frac{1}{4}$ for each point. (Here the sequence of each point corresponds to the sequence of births.)

(i) The reduced sample space consists of two elements, $\{bb, gb\}$; hence $p = \frac{1}{2}$.

(ii) The reduced sample space consists of three elements, $\{bb, bg, gb\}$; hence $p = \frac{1}{3}$.

MULTIPLICATION THEOREM FOR CONDITIONAL PROBABILITY

If we cross multiply the above equation defining conditional probability and use the fact that $A \cap E = E \cap A$, we obtain the following useful formula.

Theorem 4.2: $P(E \cap A) = P(E)\,P(A \mid E)$

This theorem can be extended by induction as follows:

Corollary 4.3: For any events $A_1, A_2, \ldots, A_n$,

$$P(A_1 \cap A_2 \cap \cdots \cap A_n)$$
$$= P(A_1)\,P(A_2 \mid A_1)\,P(A_3 \mid A_1 \cap A_2) \cdots P(A_n \mid A_1 \cap A_2 \cap \cdots \cap A_{n-1})$$

We now apply the above theorem which is called, appropriately, the *multiplication theorem.*

Example 4.3: A lot contains 12 items of which 4 are defective. Three items are drawn at random from the lot one after the other. Find the probability p that all three are nondefective.

The probability that the first item is nondefective is $\frac{8}{12}$ since 8 of 12 items are nondefective. If the first item is nondefective, then the probability that the next item is nondefective is $\frac{7}{11}$ since only 7 of the remaining 11 items are nondefective. If the first two items are nondefective, then the probability that the last item is nondefective is $\frac{6}{10}$ since only 6 of the remaining 10 items are now nondefective. Thus by the multiplication theorem,

$$p = \frac{8}{12} \cdot \frac{7}{11} \cdot \frac{6}{10} = \frac{14}{55}$$

FINITE STOCHASTIC PROCESSES AND TREE DIAGRAMS

A (finite) sequence of experiments in which each experiment has a finite number of outcomes with given probabilities is called a *(finite) stochastic process.* A convenient way of describing such a process and computing the probability of any event is by a *tree diagram* as illustrated below; the multiplication theorem of the previous section is used to compute the probability that the result represented by any given path of the tree does occur.

Example 4.4: We are given three boxes as follows:

Box I has 10 light bulbs of which 4 are defective.

Box II has 6 light bulbs of which 1 is defective.

Box III has 8 light bulbs of which 3 are defective.

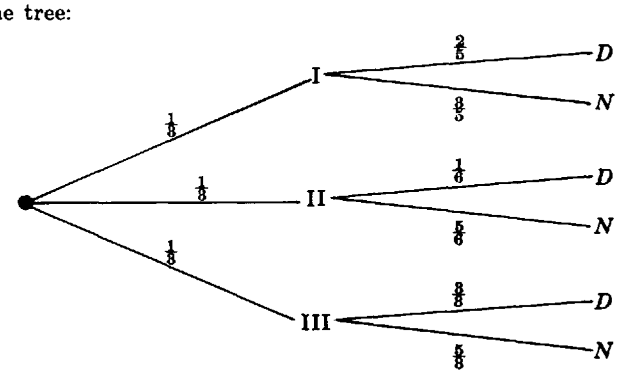
We select a box at random and then draw a bulb at random. What is the probability p that the bulb is defective?

Here we perform a sequence of two experiments:

(i) select one of the three boxes;

(ii) select a bulb which is either defective (D) or nondefective (N).

The following tree diagram describes this process and gives the probability of each branch of the tree:



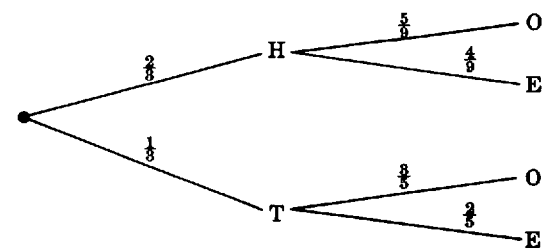
The probability that any particular path of the tree occurs is, by the multiplication theorem, the product of the probabilities of each branch of the path, e.g., the probability of selecting box I and then a defective bulb is $\frac{1}{3}\cdot\frac{2}{5}=\frac{2}{15}$.

Now since there are three mutually exclusive paths which lead to a defective bulb, the sum of the probabilities of these paths is the required probability:

$$p = \frac{1}{3}\cdot\frac{2}{5}+\frac{1}{3}\cdot\frac{1}{6}+\frac{1}{3}\cdot\frac{3}{8} = \frac{113}{360}$$

Example 4.5: A coin, weighted so that $P(\mathrm{H})=\frac{2}{3}$ and $P(\mathrm{T})=\frac{1}{3}$, is tossed. If heads appears, then a number is selected at random from the numbers 1 through 9; if tails appears, then a number is selected at random from the numbers 1 through 5. Find the probability p that an even number is selected.

The tree diagram with respective probabilities is



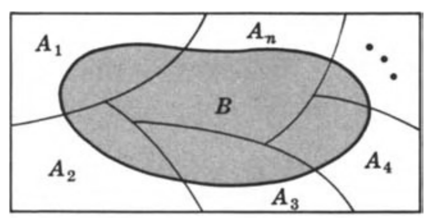
Note that the probability of selecting an even number from the numbers 1 through 9 is $\frac{4}{9}$ since there are 4 even numbers out of the 9 numbers, whereas the probability of selecting an even number from the numbers 1 through 5 is $\frac{2}{5}$ since there are 2 even numbers out of the 5 numbers. Two of the paths lead to an even number: HE and TE. Thus

$$p = P(\mathrm{E}) = \frac{2}{3}\cdot\frac{4}{9}+\frac{1}{3}\cdot\frac{2}{5} = \frac{58}{135}$$

PARTITIONS AND BAYES' THEOREM

Suppose the events $A_1, A_2, \ldots, A_n$ form a partition of a sample space S; that is, the events A_i are mutually exclusive and their union is S. Now let B be any other event. Then

$$B = S\cap B = (A_1\cup A_2\cup\cdots\cup A_n)\cap B$$
$$= (A_1\cap B)\cup(A_2\cap B)\cup\cdots\cup(A_n\cap B)$$



B is shaded.

where the $A_i\cap B$ are also mutually exclusive. Accordingly,

$$P(B) = P(A_1\cap B)+P(A_2\cap B)+\cdots+P(A_n\cap B)$$

Thus by the multiplication theorem,

$$P(B) = P(A_1)P(B\mid A_1)+P(A_2)P(B\mid A_2)+\cdots+P(A_n)P(B\mid A_n) \qquad (1)$$

On the other hand, for any i, the conditional probability of A_i given B is defined by

$$P(A_i\mid B) = \frac{P(A_i\cap B)}{P(B)}$$

In this equation we use (*1*) to replace $P(B)$ and use $P(A_i\cap B)=P(A_i)P(B\mid A_i)$ to replace $P(A_i\cap B)$, thus obtaining

Bayes' Theorem 4.4: Suppose $A_1, A_2, \ldots, A_n$ is a partition of S and B is any event. Then for any i,

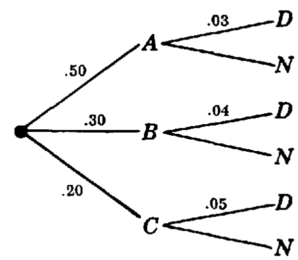
$$P(A_i\mid B) = \frac{P(A_i)P(B\mid A_i)}{P(A_1)P(B\mid A_1)+P(A_2)P(B\mid A_2)+\cdots+P(A_n)P(B\mid A_n)}$$

Example 4.6: Three machines A, B and C produce respectively 50%, 30% and 20% of the total number of items of a factory. The percentages of defective output of these machines are 3%, 4% and 5%. If an item is selected at random, find the probability that the item is defective.

Let X be the event that an item is defective. Then by (1) above,

$$P(X) = P(A)\,P(X\mid A) + P(B)\,P(X\mid B) + P(C)\,P(X\mid C)$$
$$= (.50)(.03) + (.30)(.04) + (.20)(.05)$$
$$= .037$$

Observe that we can also consider this problem as a stochastic process having the adjoining tree diagram.



Example 4.7: Consider the factory in the preceding example. Suppose an item is selected at random and is found to be defective. Find the probability that the item was produced by machine A; that is, find $P(A \mid X)$.

By Bayes' theorem,

$$P(A\mid X) = \frac{P(A)\,P(X\mid A)}{P(A)\,P(X\mid A) + P(B)\,P(X\mid B) + P(C)\,P(X\mid C)}$$
$$= \frac{(.50)(.03)}{(.50)(.03) + (.30)(.04) + (.20)(.05)} = \frac{15}{37}$$

In other words, we divide the probability of the required path by the probability of the reduced sample space, i.e. those paths which lead to a defective item.

INDEPENDENCE

An event B is said to be *independent* of an event A if the probability that B occurs is not influenced by whether A has or has not occurred. In other words, if the probability of B equals the conditional probility of B given A: $P(B) = P(B\mid A)$. Now substituting $P(B)$ for $P(B\mid A)$ in the multiplications theorem $P(A\cap B) = P(A)\,P(B\mid A)$, we obtain

$$P(A\cap B) = P(A)\,P(B)$$

We use the above equation as our formal definition of independence.

Definition: Events A and B are independent if $P(A\cap B) = P(A)\,P(B)$; otherwise they are dependent.

Example 4.8: Let a fair coin be tossed three times; we obtain the equiprobable space

$$S = \{\text{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}\}$$

Consider the events

$$A = \{\text{first toss is heads}\}, \qquad B = \{\text{second toss is heads}\}$$
$$C = \{\text{exactly two heads are tossed in a row}\}$$

Clearly A and B are independent events; this fact is verified below. On the other hand, the relationship between A and C or B and C is not obvious. We claim that A and C are independent, but that B and C are dependent. We have

$$P(A) = P(\{\text{HHH, HHT, HTH, HTT}\}) = \tfrac{4}{8} = \tfrac{1}{2}$$
$$P(B) = P(\{\text{HHH, HHT, THH, THT}\}) = \tfrac{4}{8} = \tfrac{1}{2}$$
$$P(C) = P(\{\text{HHT, THH}\}) = \tfrac{2}{8} = \tfrac{1}{4}$$

Then

$$P(A\cap B) = P(\{\text{HHH, HHT}\}) = \tfrac{1}{4}, \qquad P(A\cap C) = P(\{\text{HHT}\}) = \tfrac{1}{8},$$
$$P(B\cap C) = P(\{\text{HHT, THH}\}) = \tfrac{1}{4}$$

Accordingly,

$P(A)P(B) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4} = P(A\cap B)$, and so A and B are independent;

$P(A)P(C) = \frac{1}{2}\cdot\frac{1}{4} = \frac{1}{8} = P(A\cap C)$, and so A and C are independent;

$P(B)P(C) = \frac{1}{2}\cdot\frac{1}{4} = \frac{1}{8} \neq P(B\cap C)$, and so B and C are dependent.

Frequently, we will postulate that two events are independent, or it will be clear from the nature of the experiment that two events are independent.

Example 4.9: The probability that A hits a target is $\frac{1}{4}$ and the probability that B hits it is $\frac{2}{5}$. What is the probability that the target will be hit if A and B each shoot at the target?

We are given that $P(A) = \frac{1}{4}$ and $P(B) = \frac{2}{5}$, and we seek $P(A\cup B)$. Furthermore, the probability that A or B hits the target is not influenced by what the other does; that is, the event that A hits the target is independent of the event that B hits the target: $P(A\cap B) = P(A)P(B)$. Thus

$$P(A\cup B) = P(A) + P(B) - P(A\cap B) = P(A) + P(B) - P(A)P(B)$$
$$= \frac{1}{4} + \frac{2}{5} - \frac{1}{4}\cdot\frac{2}{5} = \frac{11}{20}$$

Three events A, B and C are *independent* if:

(i) $P(A\cap B) = P(A)P(B)$, $P(A\cap C) = P(A)P(C)$ and $P(B\cap C) = P(B)P(C)$

i.e. if the events are pairwise independent, and

(ii) $P(A\cap B\cap C) = P(A)P(B)P(C)$.

The next example shows that condition (ii) does not follow from condition (i); in other words, three events may be pairwise independent but not independent themselves.

Example 4.10: Let a pair of fair coins be tossed; here $S = \{HH, HT, TH, TT\}$ is an equiprobable space. Consider the events

$$A = \{\text{heads on the first coin}\} = \{HH, HT\}$$
$$B = \{\text{heads on the second coin}\} = \{HH, TH\}$$
$$C = \{\text{heads on exactly one coin}\} = \{HT, TH\}$$

Then $P(A) = P(B) = P(C) = \frac{2}{4} = \frac{1}{2}$ and

$$P(A\cap B) = P(\{HH\}) = \frac{1}{4}, \quad P(A\cap C) = P(\{HT\}) = \frac{1}{4}, \quad P(B\cap C) = (\{TH\}) = \frac{1}{4}$$

Thus condition (i) is satisfied, i.e., the events are pairwise independent. However, $A\cap B\cap C = \emptyset$ and so

$$P(A\cap B\cap C) = P(\emptyset) = 0 \neq P(A)P(B)P(C)$$

In other words, condition (ii) is not satisfied and so the three events are not independent.

INDEPENDENT OR REPEATED TRIALS

We have previously discussed probability spaces which were associated with an experiment repeated a finite number of times, as the tossing of a coin three times. This concept of repetition is formalized as follows:

Definition: Let S be a finite probability space. By n *independent* or *repeated trials*, we mean the probability space T consisting of ordered n-tuples of elements of S with the probability of an n-tuple defined to be the product of the probabilities of its components:

$$P((s_1, s_2, \ldots, s_n)) = P(s_1)P(s_2)\cdots P(s_n)$$

Example 4.11: Whenever three horses a, b and c race together, their respective probabilities of winning are $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{6}$. In other words, $S = \{a, b, c\}$ with $P(a) = \frac{1}{2}$, $P(b) = \frac{1}{3}$ and $P(c) = \frac{1}{6}$. If the horses race twice, then the sample space of the 2 repeated trials is

$$T = \{aa, ab, ac, ba, bb, bc, ca, cb, cc\}$$

For notational convenience, we have written ac for the ordered pair (a, c). The probability of each point in T is

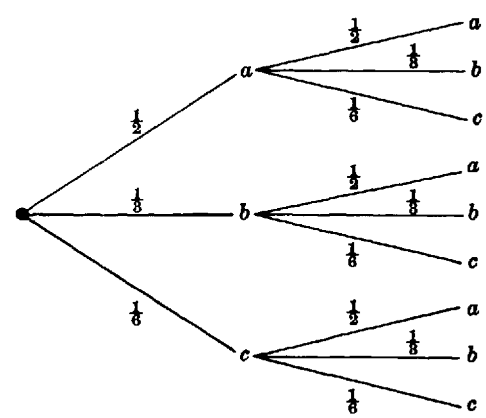
$$P(aa) = P(a)\,P(a) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4} \qquad P(ba) = \frac{1}{6} \qquad P(ca) = \frac{1}{12}$$

$$P(ab) = P(a)\,P(b) = \frac{1}{2}\cdot\frac{1}{3} = \frac{1}{6} \qquad P(bb) = \frac{1}{9} \qquad P(cb) = \frac{1}{18}$$

$$P(ac) = P(a)\,P(c) = \frac{1}{2}\cdot\frac{1}{6} = \frac{1}{12} \qquad P(bc) = \frac{1}{18} \qquad P(cc) = \frac{1}{36}$$

Thus the probability of c winning the first race and a winning the second race is $P(ca) = \frac{1}{12}$.

From another point of view, a repeated trials process is a stochastic process whose tree diagram has the following properties: (i) every branch point has the same outcomes; (ii) the probability is the same for each branch leading to the same outcome. For example, the tree diagram of the repeated trials process of the preceding experiment is as shown in the adjoining figure.



Observe that every branch point has the outcomes a, b and c, and each branch leading to outcome a has probability $\frac{1}{2}$, each branch leading to b has probability $\frac{1}{3}$, and each leading to c has probability $\frac{1}{6}$.

Solved Problems

CONDITIONAL PROBABILITY IN FINITE EQUIPROBABLE SPACES

4.1. A pair of fair dice is thrown. Find the probability p that the sum is 10 or greater if (i) a 5 appears on the first die, (ii) a 5 appears on at least one of the dice.

(i) If a 5 appears on the first die, then the reduced sample space is

$$A = \{(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)\}$$

The sum is 10 or greater on two of the six outcomes: (5, 5), (5, 6). Hence $p = \frac{2}{6} = \frac{1}{3}$.

(ii) If a 5 appears on at least one of the dice, then the reduced sample space has eleven elements:

$$B = \{(5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (1,5), (2,5), (3,5), (4,5), (6,5)\}$$

The sum is 10 or greater on three of the eleven outcomes: (5, 5), (5, 6), (6, 5). Hence $p = \frac{3}{11}$.

4.2. Three fair coins are tossed. Find the probability p that they are all heads if (i) the first coin is heads, (ii) one of the coins is heads.

The sample space has eight elements: $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$.

(i) If the first coin is heads, the reduced sample space is $A = \{HHH, HHT, HTH, HTT\}$. Since the coins are all heads in 1 of 4 cases, $p = \frac{1}{4}$.

(ii) If one of the coins is heads, the reduced sample space is $B = \{HHH, HHT, HTH, HTT, THH, THT, TTH\}$. Since the coins are all heads in 1 of 7 cases, $p = \frac{1}{7}$.

4.3. A pair of fair dice is thrown. If the two numbers appearing are different, find the probability p that (i) the sum is six, (ii) an ace appears, (iii) the sum is 4 or less.

Of the 36 ways the pair of dice can be thrown, 6 will contain the same numbers: $(1,1), (2,2), \ldots, (6,6)$. Thus the reduced sample space will consist of $36 - 6 = 30$ elements.

(i) The sum 6 can appear in 4 ways: $(1,5), (2,4), (4,2), (5,1)$. (We cannot include $(3,3)$ since the numbers are the same.) Hence $p = \frac{4}{30} = \frac{2}{15}$.

(ii) An ace can appear in 10 ways: $(1,2), (1,3), \ldots, (1,6)$ and $(2,1), (3,1), \ldots, (6,1)$. Hence $p = \frac{10}{30} = \frac{1}{3}$.

(iii) The sum of 4 or less can occur in 4 ways: $(3,1), (1,3), (2,1), (1,2)$. Thus $p = \frac{4}{30} = \frac{2}{15}$.

4.4. Two digits are selected at random from the digits 1 through 9. If the sum is even, find the probability p that both numbers are odd.

The sum is even if both numbers are even or if both numbers are odd. There are 4 even numbers $(2,4,6,8)$; hence there are $\binom{4}{2} = 6$ ways to choose two even numbers. There are 5 odd numbers $(1,3,5,7,9)$; hence there are $\binom{5}{2} = 10$ ways to choose two odd numbers. Thus there are $6 + 10 = 16$ ways to choose two numbers such that their sum is even; since 10 of these ways occur when both numbers are odd, $p = \frac{10}{16} = \frac{5}{8}$.

4.5. A man is dealt 4 spade cards from an ordinary deck of 52 cards. If he is given three more cards, find the probability p that at least one of the additional cards is also a spade.

Since he is dealt 4 spades, there are $52 - 4 = 48$ cards remaining of which $13 - 4 = 9$ are spades. There are $\binom{48}{3} = 17{,}296$ ways in which he can be dealt three more cards. Since there are $48 - 9 = 39$ cards which are not spades, there are $\binom{39}{3} = 9139$ ways he can be dealt three cards which are not spades. Thus the probability q that he is not dealt another spade is $q = \frac{9139}{17{,}296}$; hence $p = 1 - q = \frac{8157}{17{,}296}$.

4.6. Four people, called North, South, East and West, are each dealt 13 cards from an ordinary deck of 52 cards.

(i) If South has no aces, find the probability p that his partner North has exactly two aces.

(ii) If North and South together have nine hearts, find the probability p that East and West each has two hearts.

(i) There are 39 cards, including 4 aces, divided among North, East and West. There are $\binom{39}{13}$ ways that North can be dealt 13 of the 39 cards. There are $\binom{4}{2}$ ways he can be dealt 2 of the four aces, and $\binom{35}{11}$ ways he can be dealt 11 cards from the $39 - 4 = 35$ cards which are not aces. Thus

$$p = \frac{\binom{4}{2}\binom{35}{11}}{\binom{39}{13}} = \frac{6 \cdot 12 \cdot 13 \cdot 25 \cdot 26}{36 \cdot 37 \cdot 38 \cdot 39} = \frac{650}{2109}$$

(ii) There are 26 cards, including 4 hearts, divided among East and West. There are $\binom{26}{13}$ ways that, say, East can be dealt 13 cards. (We need only analyze East's 13 cards since West must have the remaining cards.) There are $\binom{4}{2}$ ways East can be dealt 2 hearts from 4 hearts, and $\binom{22}{11}$ ways he can be dealt 11 non-hearts from the $26-4=22$ non-hearts. Thus

$$p = \frac{\binom{4}{2}\binom{22}{11}}{\binom{26}{13}} = \frac{6 \cdot 12 \cdot 13 \cdot 12 \cdot 13}{23 \cdot 24 \cdot 25 \cdot 26} = \frac{234}{575}$$

MULTIPLICATION THEOREM

4.7. A class has 12 boys and 4 girls. If three students are selected at random from the class, what is the probability p that they are all boys?

The probability that the first student selected is a boy is 12/16 since there are 12 boys out of 16 students. If the first student is a boy, then the probability that the second is a boy is 11/15 since there are 11 boys left out of 15 students. Finally, if the first two students selected were boys, then the probability that the third student is a boy is 10/14 since there are 10 boys left out of 14 students. Thus, by the multiplication theorem, the probability that all three are boys is

$$p = \frac{12}{16} \cdot \frac{11}{15} \cdot \frac{10}{14} = \frac{11}{28}$$

Another Method. There are $\binom{16}{3} = 560$ ways to select 3 students of the 16 students, and $\binom{12}{3} = 220$ ways to select 3 boys out of 12 boys; hence $p = \frac{220}{560} = \frac{11}{28}$.

A Third Method. If the students are selected one after the other, then there are $16 \cdot 15 \cdot 14$ ways to select three students, and $12 \cdot 11 \cdot 10$ ways to select three boys; hence $p = \frac{12 \cdot 11 \cdot 10}{16 \cdot 15 \cdot 14} = \frac{11}{28}$.

4.8. A man is dealt 5 cards one after the other from an ordinary deck of 52 cards. What is the probability p that they are all spades?

The probability that the first card is a spade is 13/52, the second is a spade is 12/51, the third is a spade is 11/50, the fourth is a spade is 10/49, and the last is a spade is 9/48. (We assumed in each case that the previous cards were spades.) Thus $p = \frac{13}{52} \cdot \frac{12}{51} \cdot \frac{11}{50} \cdot \frac{10}{49} \cdot \frac{9}{48} = \frac{33}{66{,}640}$.

4.9. An urn contains 7 red marbles and 3 white marbles. Three marbles are drawn from the urn one after the other. Find the probability p that the first two are red and the third is white.

The probability that the first marble is red is 7/10 since there are 7 red marbles out of 10 marbles. If the first marble is red, then the probability that the second marble is red is 6/9 since there are 6 red marbles remaining out of the 9 marbles. If the first two marbles are red, then the probability that the third marble is white is 3/8 since there are 3 white marbles out of the 8 marbles in the urn. Hence by the multiplication theorem,

$$p = \frac{7}{10} \cdot \frac{6}{9} \cdot \frac{3}{8} = \frac{7}{40}$$

4.10. The students in a class are selected at random, one after the other, for an examination. Find the probability p that the boys and girls in the class alternate if (i) the class consists of 4 boys and 3 girls, (ii) the class consists of 3 boys and 3 girls.

(i) If the boys and girls are to alternate, then the first student examined must be a boy. The probability that the first is a boy is 4/7. If the first is a boy, then the probability that the second is a girl is 3/6 since there are 3 girls out of 6 students left. Continuing in this manner, we obtain the probability that the third is a boy is 3/5, the fourth is a girl is 2/4, the fifth is a boy is 2/3, the sixth is a girl is 1/2, and the last is a boy is 1/1. Thus

$$p = \frac{4}{7} \cdot \frac{3}{6} \cdot \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{2}{3} \cdot \frac{1}{2} \cdot \frac{1}{1} = \frac{1}{35}$$

(ii) There are two mutually exclusive cases: the first pupil is a boy, and the first is a girl. If the first student is a boy, then by the multiplication theorem the probability p_1 that the students alternate is

$$p_1 = \frac{3}{6}\cdot\frac{3}{5}\cdot\frac{2}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}\cdot\frac{1}{1} = \frac{1}{20}$$

If the first student is a girl, then by the multiplication theorem the probability p_2 that the students alternate is

$$p_2 = \frac{3}{6}\cdot\frac{3}{5}\cdot\frac{2}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}\cdot\frac{1}{1} = \frac{1}{20}$$

Thus $p = p_1 + p_2 = \frac{1}{20} + \frac{1}{20} = \frac{1}{10}$.

MISCELLANEOUS PROBLEMS ON CONDITIONAL PROBABILITY

4.11. In a certain college, 25% of the students failed mathematics, 15% of the students failed chemistry, and 10% of the students failed both mathematics and chemistry. A student is selected at random.

(i) If he failed chemistry, what is the probability that he failed mathematics?

(ii) If he failed mathematics, what is the probability that he failed chemistry?

(iii) What is the probability that he failed mathematics or chemistry?

Let M = {students who failed mathematics} and C = {students who failed chemistry}; then

$$P(M) = .25, \quad P(C) = .15, \quad P(M \cap C) = .10$$

(i) The probability that a student failed mathematics, given that he has failed chemistry is

$$P(M \mid C) = \frac{P(M \cap C)}{P(C)} = \frac{.10}{.15} = \frac{2}{3}$$

(ii) The probability that a student failed chemistry, given that he has failed mathematics is

$$P(C \mid M) = \frac{P(C \cap M)}{P(M)} = \frac{.10}{.25} = \frac{2}{5}$$

(iii)
$$P(M \cup C) = P(M) + P(C) - P(M \cap C) = .25 + .15 - .10 = .30 = \frac{3}{10}$$

4.12. Let A and B be events with $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$ and $P(A \cap B) = \frac{1}{4}$. Find (i) $P(A \mid B)$, (ii) $P(B \mid A)$, (iii) $P(A \cup B)$, (iv) $P(A^c \mid B^c)$, (v) $P(B^c \mid A^c)$.

(i) $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{\frac{1}{4}}{\frac{1}{3}} = \dfrac{3}{4}$ (ii) $P(B \mid A) = \dfrac{P(B \cap A)}{P(A)} = \dfrac{\frac{1}{4}}{\frac{1}{2}} = \dfrac{1}{2}$

(iii) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{4} = \dfrac{7}{12}$

(iv) First compute $P(B^c)$ and $P(A^c \cap B^c)$. $P(B^c) = 1 - P(B) = 1 - \frac{1}{3} = \frac{2}{3}$. By De Morgan's law, $(A \cup B)^c = A^c \cap B^c$; hence $P(A^c \cap B^c) = P((A \cup B)^c) = 1 - P(A \cup B) = 1 - \frac{7}{12} = \frac{5}{12}$.

Thus $P(A^c \mid B^c) = \dfrac{P(A^c \cap B^c)}{P(B^c)} = \dfrac{\frac{5}{12}}{\frac{2}{3}} = \dfrac{5}{8}$.

(v) $P(A^c) = 1 - P(A) = 1 - \frac{1}{2} = \frac{1}{2}$. Then $P(B^c \mid A^c) = \dfrac{P(B^c \cap A^c)}{P(A^c)} = \dfrac{\frac{5}{12}}{\frac{1}{2}} = \dfrac{5}{6}$.

4.13. Let A and B be events with $P(A) = \frac{3}{8}$, $P(B) = \frac{5}{8}$ and $P(A \cup B) = \frac{3}{4}$. Find $P(A \mid B)$ and $P(B \mid A)$.

First compute $P(A \cap B)$ using the formula $P(A \cup B) = P(A) + P(B) - P(A \cap B)$:

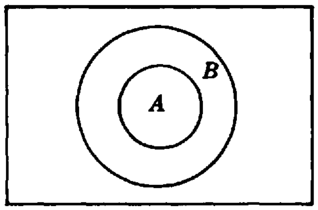
$$\frac{3}{4} = \frac{3}{8} + \frac{5}{8} - P(A \cap B) \quad \text{or} \quad P(A \cap B) = \frac{1}{4}$$

Then $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{\frac{1}{4}}{\frac{5}{8}} = \dfrac{2}{5}$ and $P(B \mid A) = \dfrac{P(B \cap A)}{P(A)} = \dfrac{\frac{1}{4}}{\frac{3}{8}} = \dfrac{2}{3}$.

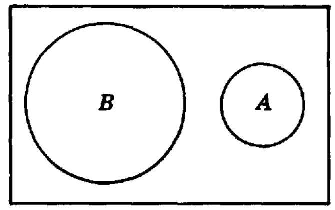
4.14. Find $P(B \mid A)$ if (i) A is a subset of B, (ii) A and B are mutually exclusive.

(i) If A is a subset of B, then whenever A occurs B must occur; hence $P(B \mid A) = 1$. Alternately, if A is a subset of B then $A \cap B = A$; hence

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A)}{P(A)} = 1$$



(i)



(ii)

(ii) If A and B are mutually exclusive, i.e. disjoint, then whenever A occurs B cannot occur; hence $P(B \mid A) = 0$. Alternately, if A and B are mutually exclusive then $A \cap B = \emptyset$; hence

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{P(\emptyset)}{P(A)} = \frac{0}{P(A)} = 0$$

4.15. Three machines A, B and C produce respectively 60%, 30% and 10% of the total number of items of a factory. The percentages of defective output of these machines are respectively 2%, 3% and 4%. An item is selected at random and is found defective. Find the probability that the item was produced by machine C.

Let $X = \{\text{defective items}\}$. We seek $P(C \mid X)$, the probability that an item is produced by machine C given that the item is defective. By Bayes' theorem,

$$P(C \mid X) = \frac{P(C)\,P(X \mid C)}{P(A)\,P(X \mid A) + P(B)\,P(X \mid B) + P(C)\,P(X \mid C)}$$

$$= \frac{(.10)(.04)}{(.60)(.02) + (.30)(.03) + (.10)(.04)} = \frac{4}{25}$$

4.16. In a certain college, 4% of the men and 1% of the women are taller than 6 feet. Furthermore, 60% of the students are women. Now if a student is selected at random and is taller than 6 feet, what is the probability that the student is a woman?

Let $A = \{\text{students taller than 6 feet}\}$. We seek $P(W \mid A)$, the probability that a student is a woman given that the student is taller than 6 feet. By Bayes' theorem,

$$P(W \mid A) = \frac{P(W)\,P(A \mid W)}{P(W)\,P(A \mid W) + P(M)\,P(A \mid M)} = \frac{(.60)(.01)}{(.60)(.01) + (.40)(.04)} = \frac{3}{11}$$

4.17. Let E be an event for which $P(E) > 0$. Show that the conditional probability function $P(* \mid E)$ satisfies the axioms of a probability space; that is,

[P_1] For any event A, $0 \leq P(A \mid E) \leq 1$.

[P_2] For the certain event S, $P(S \mid E) = 1$.

[P_3] If A and B are mutually exclusive, then $P(A \cup B \mid E) = P(A \mid E) + P(B \mid E)$.

[P_4] If $A_1, A_2, \ldots$ is a sequence of mutually exclusive events, then

$$P(A_1 \cup A_2 \cup \cdots \mid E) = P(A_1 \mid E) + P(A_2 \mid E) + \cdots$$

(i) We have $A \cap E \subset E$; hence $P(A \cap E) \leq P(E)$. Thus $P(A \mid E) = \frac{P(A \cap E)}{P(E)} \leq 1$ and is also non-negative. That is, $0 \leq P(A \mid E) \leq 1$ and so **[P_1]** holds.

(ii) We have $S \cap E = E$; hence $P(S \mid E) = \dfrac{P(S \cap E)}{P(E)} = \dfrac{P(E)}{P(E)} = 1$. Thus **[P$_2$]** holds.

(iii) If A and B are mutually exclusive events, then so are $A \cap E$ and $B \cap E$. Furthermore, $(A \cup B) \cap E = (A \cap E) \cup (B \cap E)$. Thus

$$P((A \cup B) \cap E) = P((A \cap E) \cup (B \cap E)) = P(A \cap E) + P(B \cap E)$$

and therefore

$$P(A \cup B \mid E) = \frac{P((A \cup B) \cap E)}{P(E)} = \frac{P(A \cap E) + P(B \cap E)}{P(E)}$$
$$= \frac{P(A \cap E)}{P(E)} + \frac{P(B \cap E)}{P(E)} = P(A \mid E) + P(B \mid E)$$

Hence **[P$_3$]** holds.

(iv) Similarly if $A_1, A_2, \ldots$ are mutually exclusive, then so are $A_1 \cap E, A_2 \cap E, \ldots$. Thus

$$P((A_1 \cup A_2 \cup \cdots) \cap E) = P((A_1 \cap E) \cup (A_2 \cap E) \cup \cdots) = P(A_1 \cap E) + P(A_2 \cap E) + \cdots$$

and therefore

$$P(A_1 \cup A_2 \cup \cdots \mid E) = \frac{P((A_1 \cup A_2 \cup \cdots) \cap E)}{P(E)} = \frac{P(A_1 \cap E) + P(A_2 \cap E) + \cdots}{P(E)}$$
$$= \frac{P(A_1 \cap E)}{P(E)} + \frac{P(A_2 \cap E)}{P(E)} + \cdots = P(A_1 \mid E) + P(A_2 \mid E) + \cdots$$

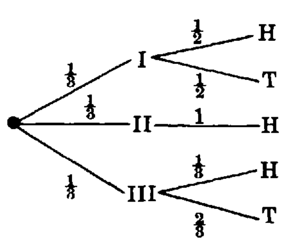
That is, **[P$_4$]** holds.

FINITE STOCHASTIC PROCESSES

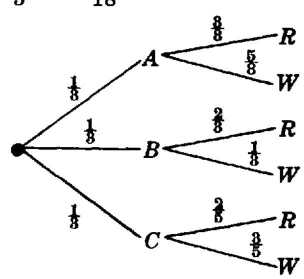
4.18. A box contains three coins; one coin is fair, one coin is two-headed, and one coin is weighted so that the probability of heads appearing is $\frac{1}{3}$. A coin is selected at random and tossed. Find the probability p that heads appears.

Construct the tree diagram as shown in Figure (*a*) below. Note that I refers to the fair coin, II to the two-headed coin, and III to the weighted coin. Now heads appears along three of the paths; hence

$$p = \frac{1}{3}\cdot\frac{1}{2} + \frac{1}{3}\cdot 1 + \frac{1}{3}\cdot\frac{1}{3} = \frac{11}{18}$$



(*a*)



(*b*)

4.19. We are given three urns as follows:

Urn A contains 3 red and 5 white marbles.
Urn B contains 2 red and 1 white marble.
Urn C contains 2 red and 3 white marbles.

An urn is selected at random and a marble is drawn from the urn. If the marble is red, what is the probability that it came from urn A?

Construct the tree diagram as shown in Figure (*b*) above.

We seek the probability that A was selected, given that the marble is red; that is, $P(A \mid R)$. In order to find $P(A \mid R)$, it is necessary first to compute $P(A \cap R)$ and $P(R)$.

The probability that urn A is selected and a red marble drawn is $\frac{1}{3}\cdot\frac{3}{8} = \frac{1}{8}$; that is, $P(A \cap R) = \frac{1}{8}$. Since there are three paths leading to a red marble, $P(R) = \frac{1}{3}\cdot\frac{3}{8} + \frac{1}{3}\cdot\frac{2}{3} + \frac{1}{3}\cdot\frac{2}{5} = \frac{173}{360}$. Thus

$$P(A \mid R) = \frac{P(A \cap R)}{P(R)} = \frac{\frac{1}{8}}{\frac{173}{360}} = \frac{45}{173}$$

Alternately, by Bayes' theorem,

$$P(A \mid R) = \frac{P(A)\,P(R \mid A)}{P(A)\,P(R \mid A) + P(B)\,P(R \mid B) + P(C)\,P(R \mid C)}$$

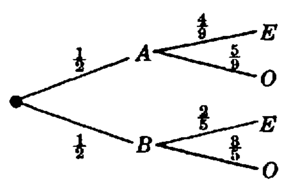
$$= \frac{\frac{1}{3}\cdot\frac{3}{8}}{\frac{1}{3}\cdot\frac{3}{8} + \frac{1}{3}\cdot\frac{2}{3} + \frac{1}{3}\cdot\frac{2}{5}} = \frac{45}{173}$$

4.20. Box A contains nine cards numbered 1 through 9, and box B contains five cards numbered 1 through 5. A box is chosen at random and a card drawn. If the number is even, find the probability that the card came from box A.

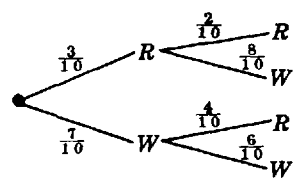
The tree diagram of the process is shown in Figure (*a*) below.

We seek $P(A \mid E)$, the probability that A was selected, given that the number is even. The probability that box A and an even number is drawn is $\frac{1}{2}\cdot\frac{4}{9} = \frac{2}{9}$; that is, $P(A \cap E) = \frac{2}{9}$. Since there are two paths which lead to an even number, $P(E) = \frac{1}{2}\cdot\frac{4}{9} + \frac{1}{2}\cdot\frac{2}{5} = \frac{19}{45}$. Thus

$$P(A \mid E) = \frac{P(A \cap E)}{P(E)} = \frac{\frac{2}{9}}{\frac{19}{45}} = \frac{10}{19}$$



(*a*)



(*b*)

4.21. An urn contains 3 red marbles and 7 white marbles. A marble is drawn from the urn and a marble of the other color is then put into the urn. A second marble is drawn from the urn.

(i) Find the probability p that the second marble is red.

(ii) If both marbles were of the same color, what is the probability p that they were both white?

Construct the tree diagram as shown in Figure (*b*) above.

(i) Two paths of the tree lead to a red marble: $p = \frac{3}{10}\cdot\frac{2}{10} + \frac{7}{10}\cdot\frac{4}{10} = \frac{17}{50}$.

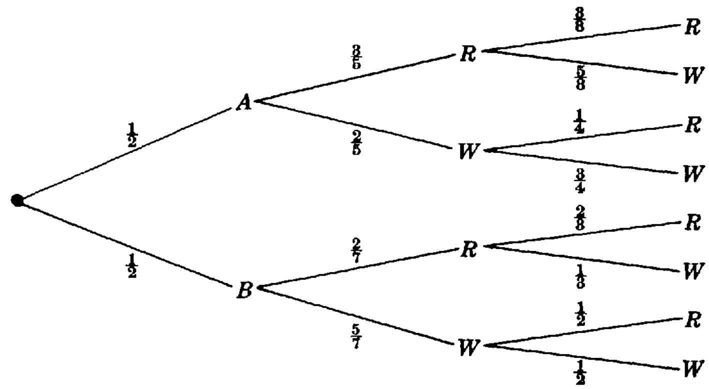
(ii) The probability that both marbles were white is $\frac{7}{10}\cdot\frac{6}{10} = \frac{21}{50}$. The probability that both marbles were of the same color, i.e. the probability of the reduced sample space, is $\frac{3}{10}\cdot\frac{2}{10} + \frac{7}{10}\cdot\frac{6}{10} = \frac{12}{25}$. Hence the conditional probability $p = \frac{21}{50}/\frac{12}{25} = \frac{7}{8}$.

4.22. We are given two urns as follows:

Urn A contains 3 red and 2 white marbles.
Urn B contains 2 red and 5 white marbles.

An urn is selected at random; a marble is drawn and put into the other urn; then a marble is drawn from the second urn. Find the probability p that both marbles drawn are of the same color.

Construct the following tree diagram:



Note that if urn A is selected and a red marble drawn and put into urn B, then urn B has 3 red marbles and 5 white marbles.

Since there are four paths which lead to two marbles of the same color,

$$p = \frac{1}{2}\cdot\frac{3}{5}\cdot\frac{3}{8} + \frac{1}{2}\cdot\frac{2}{5}\cdot\frac{3}{4} + \frac{1}{2}\cdot\frac{2}{7}\cdot\frac{2}{3} + \frac{1}{2}\cdot\frac{5}{7}\cdot\frac{1}{2} = \frac{901}{1680}$$

INDEPENDENCE

4.23. Let A = event that a family has children of both sexes, and let B = event that a family has at most one boy. (i) Show that A and B are independent events if a family has three children. (ii) Show that A and B are dependent events if a family has two children.

(i) We have the equiprobable space $S = \{bbb, bbg, bgb, bgg, gbb, gbg, ggb, ggg\}$. Here

$$A = \{bbg, bgb, bgg, gbb, gbg, ggb\} \quad \text{and so} \quad P(A) = \frac{6}{8} = \frac{3}{4}$$

$$B = \{bgg, gbg, ggb, ggg\} \quad \text{and so} \quad P(B) = \frac{4}{8} = \frac{1}{2}$$

$$A \cap B = \{bgg, gbg, ggb\} \quad \text{and so} \quad P(A \cap B) = \frac{3}{8}$$

Since $P(A)\,P(B) = \frac{3}{4}\cdot\frac{1}{2} = \frac{3}{8} = P(A \cap B)$, A and B are independent.

(ii) We have the equiprobable space $S = \{bb, bg, gb, gg\}$. Here

$$A = \{bg, gb\} \quad \text{and so} \quad P(A) = \frac{1}{2}$$

$$B = \{bg, gb, gg\} \quad \text{and so} \quad P(B) = \frac{3}{4}$$

$$A \cap B = \{bg, gb\} \quad \text{and so} \quad P(A \cap B) = \frac{1}{2}$$

Since $P(A)\,P(B) \neq P(A \cap B)$, A and B are dependent.

4.24. Prove: If A and B are independent events, then A^c and B^c are independent events.

$$\begin{aligned} P(A^c \cap B^c) &= P((A \cup B)^c) = 1 - P(A \cup B) = 1 - P(A) - P(B) + P(A \cap B) \\ &= 1 - P(A) - P(B) + P(A)\,P(B) = [1 - P(A)][1 - P(B)] = P(A^c)\,P(B^c) \end{aligned}$$

4.25. The probability that a man will live 10 more years is $\frac{1}{4}$, and the probability that his wife will live 10 more years is $\frac{1}{3}$. Find the probability that (i) both will be alive in 10 years, (ii) at least one will be alive in 10 years, (iii) neither will be alive in 10 years, (iv) only the wife will be alive in 10 years.

Let A = event that the man is alive in 10 years, and B = event that his wife is alive in 10 years; then $P(A) = \frac{1}{4}$ and $P(B) = \frac{1}{3}$.

(i) We seek $P(A \cap B)$. Since A and B are independent, $P(A \cap B) = P(A)\,P(B) = \frac{1}{4}\cdot\frac{1}{3} = \frac{1}{12}$.

(ii) We seek $P(A \cup B)$. $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{4} + \frac{1}{3} - \frac{1}{12} = \frac{1}{2}$.

(iii) We seek $P(A^c \cap B^c)$. Now $P(A^c) = 1 - P(A) = 1 - \frac{1}{4} = \frac{3}{4}$ and $P(B^c) = 1 - P(B) = 1 - \frac{1}{3} = \frac{2}{3}$. Furthermore, since A^c and B^c are independent, $P(A^c \cap B^c) = P(A^c)\,P(B^c) = \frac{3}{4} \cdot \frac{2}{3} = \frac{1}{2}$.

Alternately, since $(A \cup B)^c = A^c \cap B^c$, $P(A^c \cap B^c) = P((A \cup B)^c) = 1 - P(A \cup B) = 1 - \frac{1}{2} = \frac{1}{2}$.

(iv) We seek $P(A^c \cap B)$. Since $P(A^c) = 1 - P(A) = \frac{3}{4}$ and A^c and B are independent (see Problem 4.56), $P(A^c \cap B) = P(A^c)\,P(B) = \frac{1}{4}$.

4.26. Box A contains 8 items of which 3 are defective, and box B contains 5 items of which 2 are defective. An item is drawn at random from each box.

(i) What is the probability p that both items are nondefective?

(ii) What is the probability p that one item is defective and one not?

(iii) If one item is defective and one is not, what is the probability p that the defective item came from box A?

(i) The probability of choosing a nondefective item from A is $\frac{5}{8}$ and from B is $\frac{3}{5}$. Since the events are independent, $p = \frac{5}{8} \cdot \frac{3}{5} = \frac{3}{8}$.

(ii) **Method 1.** The probability of choosing two defective items is $\frac{3}{8} \cdot \frac{2}{5} = \frac{3}{20}$. From (i) the probability that both are nondefective is $\frac{3}{8}$. Hence $p = 1 - \frac{3}{8} - \frac{3}{20} = \frac{19}{40}$.

Method 2. The probability p_1 of choosing a defective item from A and a nondefective item from B is $\frac{3}{8} \cdot \frac{3}{5} = \frac{9}{40}$. The probability p_2 of choosing a nondefective item from A and a defective item from B is $\frac{5}{8} \cdot \frac{2}{5} = \frac{1}{4}$. Hence $p = p_1 + p_2 = \frac{9}{40} + \frac{1}{4} = \frac{19}{40}$.

(iii) Consider the events $X = \{\text{defective item from } A\}$ and $Y = \{\text{one item is defective and one nondefective}\}$. We seek $P(X \mid Y)$. By (ii), $P(X \cap Y) = p_1 = \frac{9}{40}$ and $P(Y) = \frac{19}{40}$. Hence

$$p = P(X \mid Y) = \frac{P(X \cap Y)}{P(Y)} = \frac{\frac{9}{40}}{\frac{19}{40}} = \frac{9}{19}$$

4.27. The probabilities that three men hit a target are respectively $\frac{1}{6}$, $\frac{1}{4}$ and $\frac{1}{3}$. Each shoots once at the target. (i) Find the probability p that exactly one of them hits the target. (ii) If only one hit the target, what is the probability that it was the first man?

Consider the events $A = \{\text{first man hits the target}\}$, $B = \{\text{second man hits the target}\}$, and $C = \{\text{third man hits the target}\}$; then $P(A) = \frac{1}{6}$, $P(B) = \frac{1}{4}$ and $P(C) = \frac{1}{3}$. The three events are independent, and $P(A^c) = \frac{5}{6}$, $P(B^c) = \frac{3}{4}$, $P(C^c) = \frac{2}{3}$.

(i) Let $E = \{\text{exactly one man hits the target}\}$. Then

$$E = (A \cap B^c \cap C^c) \cup (A^c \cap B \cap C^c) \cup (A^c \cap B^c \cap C)$$

In other words, if only one hit the target, then it was either only the first man, $A \cap B^c \cap C^c$, or only the second man, $A^c \cap B \cap C^c$, or only the third man, $A^c \cap B^c \cap C$. Since the three events are mutually exclusive, we obtain (using Problem 4.62)

$$\begin{aligned} p = P(E) &= P(A \cap B^c \cap C^c) + P(A^c \cap B \cap C^c) + P(A^c \cap B^c \cap C) \\ &= P(A)\,P(B^c)\,P(C^c) + P(A^c)\,P(B)\,P(C^c) + P(A^c)\,P(B^c)\,P(C) \\ &= \frac{1}{6} \cdot \frac{3}{4} \cdot \frac{2}{3} + \frac{5}{6} \cdot \frac{1}{4} \cdot \frac{2}{3} + \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{3} = \frac{1}{12} + \frac{5}{36} + \frac{5}{24} = \frac{31}{72} \end{aligned}$$

(ii) We seek $P(A \mid E)$, the probability that the first man hit the target given that only one man hit the target. Now $A \cap E = A \cap B^c \cap C^c$ is the event that only the first man hit the target. By (i), $P(A \cap E) = P(A \cap B^c \cap C^c) = \frac{1}{12}$ and $P(E) = \frac{31}{72}$; hence

$$P(A \mid E) = \frac{P(A \cap E)}{P(E)} = \frac{\frac{1}{12}}{\frac{31}{72}} = \frac{6}{31}$$

INDEPENDENT TRIALS

4.28. A certain type of missile hits its target with probability .3. How many missiles should be fired so that there is at least an 80% probability of hitting a target?

The probability of a missile missing its target is .7; hence the probability that n missiles miss a target is $(.7)^n$. Thus we seek the smallest n for which

$$1-(.7)^n > .8 \quad \text{or equivalently} \quad (.7)^n < .2$$

Compute: $(.7)^1 = .7$, $(.7)^2 = .49$, $(.7)^3 = .343$, $(.7)^4 = .2401$, $(.7)^5 = .16807$. Thus at least 5 missiles should be fired.

4.29. A certain soccer team wins (W) with probability .6, loses (L) with probability .3 and ties (T) with probability .1. The team plays three games over the weekend. (i) Determine the elements of the event A that the team wins at least twice and doesn't lose; and find $P(A)$. (ii) Determine the elements of the event B that the team wins, loses and ties; and find $P(B)$.

(i) A consists of all ordered triples with at least 2 W's and no L's. Thus

$$A = \{\text{WWW, WWT, WTW, TWW}\}$$

Furthermore,
$$\begin{aligned} P(A) &= P(\text{WWW}) + P(\text{WWT}) + P(\text{WTW}) + P(\text{TWW}) \\ &= (.6)(.6)(.6) + (.6)(.6)(.1) + (.6)(.1)(.6) + (.1)(.6)(.6) \\ &= .216 + .036 + .036 + .036 = .324 \end{aligned}$$

(ii) Here $B = \{\text{WLT, WTL, LWT, LTW, TWL, TLW}\}$. Since each element of B has probability $(.6)(.3)(.1) = .018$, $P(B) = 6(.018) = .108$.

4.30. Let S be a finite probability space and let T be the probability space of n independent trials in S. Show that T is well defined; that is, show (i) the probability of each element of T is nonnegative and (ii) the sum of their probabilities is 1.

If $S = \{a_1, \ldots, a_r\}$, then T can be represented by

$$T = \{a_{i_1} \cdots a_{i_n} : i_1, \ldots, i_n = 1, \ldots, r\}$$

Since $P(a_i) \geq 0$, we have

$$P(a_{i_1} \cdots a_{i_n}) = P(a_{i_1}) \cdots P(a_{i_n}) \geq 0$$

for a typical element $a_{i_1} \cdots a_{i_n}$ in T, which proves (i)

We prove (ii) by induction on n. It is obviously true for $n = 1$. Therefore we consider $n > 1$ and assume (ii) has been proved for $n - 1$. Then

$$\sum_{i_1,\ldots,i_n=1}^{r} P(a_{i_1} \cdots a_{i_n}) = \sum_{i_1,\ldots,i_n=1}^{r} P(a_{i_1}) \cdots P(a_{i_n}) = \sum_{i_1,\ldots,i_{n-1}=1}^{r} P(a_{i_1}) \cdots P(a_{i_{n-1}}) \sum_{i_n=1}^{r} P(a_{i_n})$$

$$= \sum_{i_1,\ldots,i_{n-1}=1}^{r} P(a_{i_1}) \cdots P(a_{i_{n-1}}) = \sum_{i_1,\ldots,i_{n-1}=1}^{r} P(a_{i_1} \cdots a_{i_{n-1}}) = 1$$

by the inductive hypothesis, which proves (ii) for n.

Supplementary Problems

CONDITIONAL PROBABILITY

4.31. A die is tossed. If the number is odd, what is the probability that it is prime?

4.32. Three fair coins are tossed. If both heads and tails appear, determine the probability that exactly one head appears.

4.33. A pair of dice is tossed. If the numbers appearing are different, find the probability that the sum is even.

4.34. A man is dealt 5 red cards from an ordinary deck of 52 cards. What is the probability that they are all of the same suit, i.e. hearts or diamonds?

4.35. A man is dealt 3 spade cards from an ordinary deck of 52 cards. If he is given four more cards, determine the probability that at least two of the additional cards are also spades.

4.36. Two different digits are selected at random from the digits 1 through 9.

(i) If the sum is odd, what is the probability that 2 is one of the numbers selected?

(ii) If 2 is one of the digits selected, what is the probability that the sum is odd?

4.37. Four persons, called North, South, East and West, are each dealt 13 cards from an ordinary deck of 52 cards.

(i) If South has exactly one ace, what is the probability that his partner North has the other three aces?

(ii) If North and South together have 10 hearts, what is the probability that either East or West has the other 3 hearts?

4.38. A class has 10 boys and 5 girls. Three students are selected from the class at random, one after the other. Find the probability that (i) the first two are boys and the third is a girl, (ii) the first and third are boys and the second is a girl, (iii) the first and third are of the same sex, and the second is of the opposite sex.

4.39. In the preceding problem, if the first and third students selected are of the same sex and the second student is of the opposite sex, what is the probability that the second student is a girl?

4.40. In a certain town, 40% of the people have brown hair, 25% have brown eyes, and 15% have both brown hair and brown eyes. A person is selected at random from the town.

(i) If he has brown hair, what is the probability that he also has brown eyes?

(ii) If he has brown eyes, what is the probability that he does not have brown hair?

(iii) What is the probability that he has neither brown hair nor brown eyes?

4.41. Let A and B be events with $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{4}$ and $P(A \cup B) = \frac{1}{2}$. Find (i) $P(A \mid B)$, (ii) $P(B \mid A)$, (iii) $P(A \cap B^c)$, (iv) $P(A \mid B^c)$.

4.42. Let $S = \{a, b, c, d, e, f\}$ with $P(a) = \frac{1}{16}$, $P(b) = \frac{1}{16}$, $P(c) = \frac{1}{8}$, $P(d) = \frac{3}{16}$, $P(e) = \frac{1}{4}$ and $P(f) = \frac{5}{16}$. Let $A = \{a, c, e\}$, $B = \{c, d, e, f\}$ and $C = \{b, c, f\}$. Find (i) $P(A \mid B)$, (ii) $P(B \mid C)$, (iii) $P(C \mid A^c)$, (iv) $P(A^c \mid C)$.

4.43. In a certain college, 25% of the boys and 10% of the girls are studying mathematics. The girls constitute 60% of the student body. If a student is selected at random and is studying mathematics, determine the probability that the student is a girl.

FINITE STOCHASTIC PROCESSES

4.44. We are given two urns as follows:

Urn A contains 5 red marbles, 3 white marbles and 8 blue marbles.

Urn B contains 3 red marbles and 5 white marbles.

A fair die is tossed; if 3 or 6 appears, a marble is chosen from B, otherwise a marble is chosen from A. Find the probability that (i) a red marble is chosen, (ii) a white marble is chosen, (iii) a blue marble is chosen.

4.45. Refer to the preceding problem. (i) If a red marble is chosen, what is the probability that it came from urn A? (ii) If a white marble is chosen, what is the probability that a 5 appeared on the die?

4.46. An urn contains 5 red marbles and 3 white marbles. A marble is selected at random from the urn, discarded, and two marbles of the other color are put into the urn. A second marble is then selected from the urn. Find the probability that (i) the second marble is red, (ii) both marbles are of the same color.

4.47. Refer to the preceding problem. (i) If the second marble is red, what is the probability that the first marble is red? (ii) If both marbles are of the same color, what is the probability that they are both white?

4.48. A box contains three coins, two of them fair and one two-headed. A coin is selected at random and tossed twice. If heads appears both times, what is the probability that the coin is two-headed?

4.49. We are given two urns as follows:

Urn A contains 5 red marbles and 3 white marbles.

Urn B contains 1 red marble and 2 white marbles.

A fair die is tossed; if a 3 or 6 appears, a marble is drawn from B and put into A and then a marble is drawn from A; otherwise, a marble is drawn from A and put into B and then a marble is drawn from B.

(i) What is the probability that both marbles are red?

(ii) What is the probability that both marbles are white?

4.50. Box A contains nine cards numbered 1 through 9, and box B contains five cards numbered 1 through 5. A box is chosen at random and a card drawn; if the card shows an even number, another card is drawn from the same box; if the card shows an odd number, a card is drawn from the other box.

(i) What is the probability that both cards show even numbers?

(ii) If both cards show even numbers, what is the probability that they come from box A?

(iii) What is the probability that both cards show odd numbers?

4.51. A box contains a fair coin and a two-headed coin. A coin is selected at random and tossed. If heads appears, the other coin is tossed; if tails appears, the same coin is tossed.

(i) Find the probability that heads appears on the second toss.

(ii) If heads appeared on the second toss, find the probability that it also appeared on the first toss.

4.52. A box contains three coins, two of them fair and one two-headed. A coin is selected at random and tossed. If heads appears the coin is tossed again; if tails appears, then another coin is selected from the two remaining coins and tossed.

(i) Find the probability that heads appears twice.

(ii) If the same coin is tossed twice, find the probability that it is the two-headed coin.

(iii) Find the probability that tails appears twice.

4.53. Urn A contains x red marbles and y white marbles, and urn B contains z red marbles and v white marbles.

(i) If an urn is selected at random and a marble drawn, what is the probability that the marble is red?

(ii) If a marble is drawn from urn A and put into urn B and then a marble is drawn from urn B, what is the probability that the second marble is red?

4.54. A box contains 5 radio tubes of which 2 are defective. The tubes are tested one after the other until the 2 defective tubes are discovered. What is the probability that the process stopped on the (i) second test, (ii) third test?

4.55. Refer to the preceding problem. If the process stopped on the third test, what is the probability that the first tube is nondefective?

INDEPENDENCE

4.56. Prove: If A and B are independent, then A and B^c are independent and A^c and B are independent.

4.57. Let A and B be events with $P(A) = \frac{1}{4}$, $P(A \cup B) = \frac{1}{3}$ and $P(B) = p$. (i) Find p if A and B are mutually exclusive. (ii) Find p if A and B are independent. (iii) Find p if A is a subset of B.

4.58. Urn A contains 5 red marbles and 3 white marbles, and urn B contains 2 red marbles and 6 white marbles.

(i) If a marble is drawn from each urn, what is the probability that they are both of the same color?

(ii) If two marbles are drawn from each urn, what is the probability that all four marbles are of the same color?

4.59. Let three fair coins be tossed. Let $A = \{\text{all heads or all tails}\}$, $B = \{\text{at least two heads}\}$ and $C = \{\text{at most two heads}\}$. Of the pairs (A,B), (A,C) and (B,C), which are independent and which are dependent?

4.60. The probability that A hits a target is $\frac{1}{4}$ and the probability that B hits a target is $\frac{1}{3}$.

(i) If each fires twice, what is the probability that the target will be hit at least once?

(ii) If each fires once and the target is hit only once, what is the probability that A hit the target?

(iii) If A can fire only twice, how many times must B fire so that there is at least a 90% probability that the target will be hit?

4.61. Let A and B be independent events with $P(A) = \frac{1}{2}$ and $P(A \cup B) = \frac{2}{3}$. Find (i) $P(B)$, (ii) $P(A \mid B)$, (iii) $P(B^c \mid A)$.

4.62. Suppose A, B, C are independent events. Show that any of the combinations

$$A^c, B, C;\ A, B^c, C;\ \ldots;\ A^c, B^c, C;\ \ldots;\ A^c, B^c, C^c$$

are also independent. Furthermore, show that A and $B \cup C$ are independent; and so forth.

INDEPENDENT TRIALS

4.63. A rifleman hits (H) his target with probability .4, and hence misses (M) with probability .6. He fires four times. (i) Determine the elements of the event A that the man hits the target exactly twice; and find $P(A)$. (ii) Find the probability that the man hits the target at least once.

4.64. A team wins (W) with probability .5, loses (L) with probability .3 and ties (T) with probability .2. The team plays twice. (i) Determine the sample space S and the probabilities of the elementary events. (ii) Find the probability that the team wins at least once.

4.65. Consider a countably infinite probability space $S = \{a_1, a_2, \ldots\}$. Let

$$T = S^n = \{(s_1, s_2, \ldots, s_n) : s_i \in S\}$$

and let

$$P(s_1, s_2, \ldots, s_n) = P(s_1)\,P(s_2)\cdots P(s_n)$$

Show that T is also a countably infinite probability space. (This generalizes the definition (page 58) of independent trials to a countably infinite space.)

Answers to Supplementary Problems

4.31. $\frac{2}{3}$

4.32. $\frac{1}{2}$

4.33. $\frac{2}{5}$

4.34. $\dfrac{2\binom{13}{5}}{\binom{26}{5}} = \dfrac{9}{230}$

4.35. $1 - \dfrac{\binom{39}{4}}{\binom{49}{4}} - \dfrac{10\binom{39}{3}}{\binom{49}{4}}$

4.36. (i) $\frac{1}{4}$, (ii) $\frac{5}{8}$

4.37. (i) $\dfrac{\binom{36}{10}}{\binom{39}{13}} = \dfrac{22}{703}$ (ii) $\dfrac{2\binom{23}{10}}{\binom{26}{13}} = \dfrac{11}{50}$

4.38. (i) $\frac{10}{15} \cdot \frac{9}{14} \cdot \frac{5}{13} = \frac{15}{91}$

(ii) $\frac{10}{15} \cdot \frac{5}{14} \cdot \frac{9}{13} = \frac{15}{91}$

(iii) $\frac{15}{91} + \frac{20}{273} = \frac{5}{21}$

4.39. $\dfrac{\frac{15}{91}}{\frac{5}{21}} = \frac{9}{13}$

4.40. (i) $\frac{3}{8}$, (ii) $\frac{2}{5}$, (iii) $\frac{1}{2}$

4.41. (i) $\frac{1}{3}$, (ii) $\frac{1}{4}$, (iii) $\frac{1}{4}$, (iv) $\frac{1}{3}$

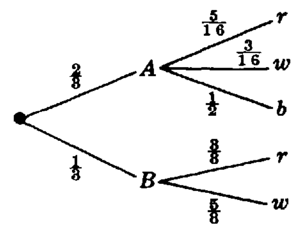
4.42. (i) $\frac{3}{7}$, (ii) $\frac{7}{8}$, (iii) $\frac{2}{3}$, (iv) $\frac{3}{4}$

4.43. $\frac{3}{8}$

4.44. (i) $\frac{1}{3}$

(ii) $\frac{1}{3}$

(iii) $\frac{1}{3}$

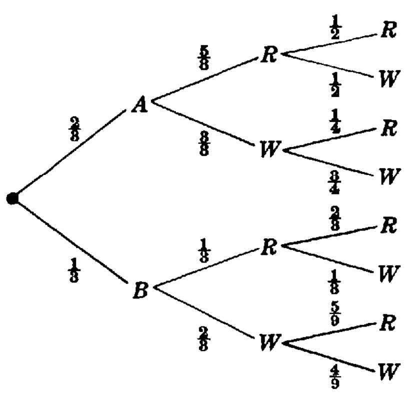


4.45. (i) $\frac{5}{8}$, (ii) $\frac{3}{32}$

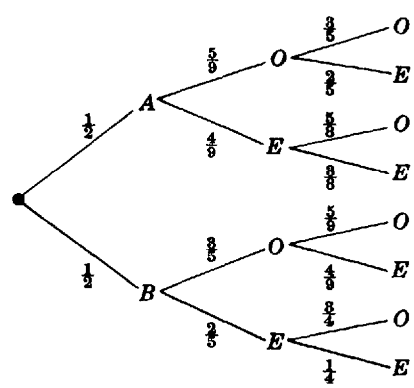
4.46. (i) $\frac{41}{72}$, (ii) $\frac{13}{36}$

4.47. (i) $\frac{20}{41}$, (ii) $\frac{3}{13}$

4.48. $\frac{2}{3}$



Tree diagram for Problem 4.49



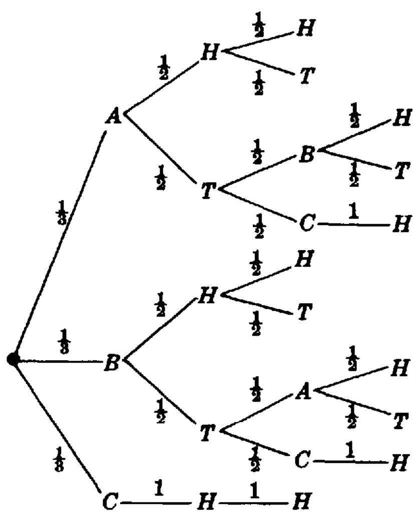
Tree diagram for Problem 4.50

4.49. (i) $\frac{5}{24} + \frac{2}{27} = \frac{61}{216}$, (ii) $\frac{3}{16} + \frac{8}{81} = \frac{371}{1296}$

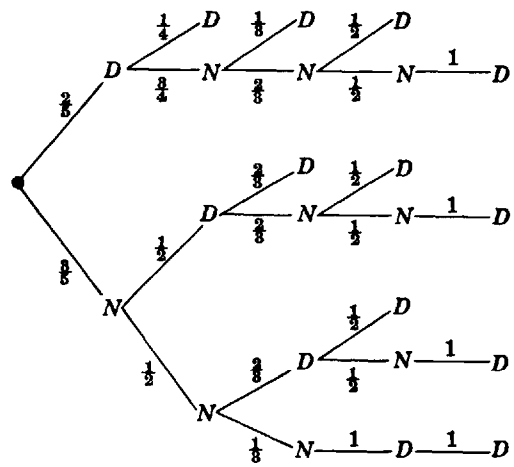
4.50. (i) $\frac{1}{12} + \frac{1}{20} = \frac{2}{15}$, (ii) $\dfrac{\frac{1}{12}}{\frac{2}{15}} = \frac{5}{8}$, (iii) $\frac{1}{6} + \frac{1}{6} = \frac{1}{3}$

4.51. (i) $\frac{5}{8}$, (ii) $\frac{4}{5}$

4.52. (i) $\frac{1}{12} + \frac{1}{12} + \frac{1}{3} = \frac{1}{2}$, (ii) $\frac{1}{2}$, (iii) $\frac{1}{12}$



Tree diagram for Problem 4.52



Tree diagram for Problem 4.54

4.53. (i) $\frac{1}{2}\left(\frac{x}{x+y} + \frac{z}{z+v}\right)$, (ii) $\frac{xz+x+yz}{(x+y)(z+v+1)}$

4.54. (i) $\frac{1}{10}$, (ii) $\frac{3}{10}$; we must include the case where the three nondefective tubes appear first, since the last two tubes must then be the defective ones.

4.55. $\frac{2}{3}$

4.57. (i) $\frac{1}{12}$, (ii) $\frac{1}{9}$, (iii) $\frac{1}{3}$

4.58. (i) $\frac{7}{16}$, (ii) $\frac{55}{784}$

4.59. Only A and B are independent.

4.60. (i) $\frac{3}{4}$, (ii) $\frac{3}{5}$, (iii) 5

4.61. (i) $\frac{1}{3}$, (ii) $\frac{1}{2}$, (iii) $\frac{2}{3}$

4.63. (i) A = {HHMM, HMHM, HMMH, MHHM, MHMH, MMHH}, $P(A) = .3456$
(ii) $1 - (.6)^4 = .8704$

4.64. (i) S = {WW, WL, WT, LW, LL, LT, TW, TL, TT}
(ii) .75