GRADE 12 APPLIED
CLASS NOTES

PROBABILITY
&
PERMUTATIONS AND COMBINATIONS

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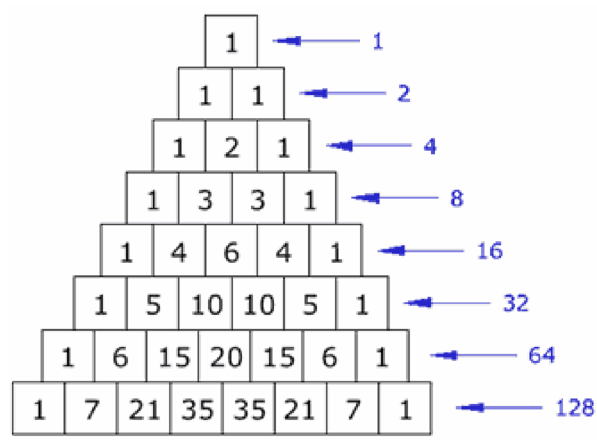
(adapted from former 2005 Manitoba Distance Learning Website)



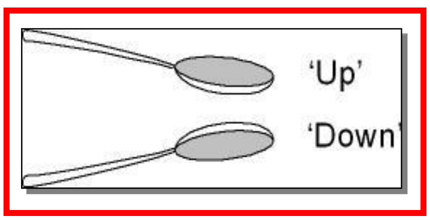
Gr12App_A_Probability_Notes

Probability

1. Lesson 1 – Experimental and Theoretical Probability

1.1 Definition of Probability

1. When a teaspoon falls to the floor, it always comes to rest "right side up" or "up side down."



2. In an '*experiment*' to determine how likely it is for the spoon to land "up" or "down," the spoon was dropped to the floor on 40 '*trials*', and the results were as follows:

Outcome	Number of Favourable Outcomes
'Up'	26
'Down'	14

What is the probability from this experiment that the spoon comes to rest '***up***'? We are not sure what will happen on any *particular* trial, but we can predict what will happen *after many* trials.

3. Before we answer this question, we need to identify and define some words associated with ***probability***. We say that the spoon experiment tested **random outcomes** because we had no way of knowing which way the spoon was going to land. A random experiment is an experiment of a certain number of trials which can result in different outcomes, and for which the outcome is unknown in advance.

4. The result of one **trial** (dropping the spoon once) of an **experiment** (dropping the spoon many times) is called an **outcome**. For example, one *outcome* is the spoon landing "**up**." **Probability** may be defined as follows:

$$Probability\ of\ a\ favouable\ outcome = \frac{number\ of\ favourable\ trials}{total\ number\ of\ trials}$$

Of course mathematicians just say: "let the desired outcome be called 'A' " then " $\boldsymbol{Prob(A) = \frac{n(A)}{Total}}$ "
where n(A) means the number of times outcome A happens.

Therefore, the probability of the spoon landing 'right side up' would be:

$$P(Right\,Side\,Up) = \frac{26}{40} = \frac{13}{20} = 0.65 = 65\%$$

'Odds' are another way to measure chance. Not quite the same as the probability

5. We normally write probabilities as fractions reduced to lowest terms, in decimal form or as percentages.

6. Here are some practice conversions between the different forms of expressing probability; make sure you know how to convert between the different representations:

Fraction	Decimal	Percent
3/8	0.375	37.5%
1/10		
4/5		
9/52		
	0.75	
	1.00	
	0.95	
		35%
		18%
	0.1666667	

1.2. Basic Properties of Probabilities

7. In the previous experiment when the spoon was dropped, the spoon never came to rest vertically, and it does not seem reasonable that it ever would do so after a zillion trials. We say the spoon has no chance, or zero probability, of landing vertically. Therefore,

$$Prob\ (vertical) = \frac{0\ vertical\ times}{1\ zillion\ trials} = 0$$

The point being that it is possible mathematically to have a probability of zero.

(*Although a 'purist' or philosopher might argue there is no such thing as zero probability, in the history of the universe surely one spoon will stand up vertically*)

The spoon always came to rest 'up' or 'down', and so we say that landing in one of these two ways was a certainty, and that the probability of landing in either one of these two ways was 100%. Therefore,

$$\boldsymbol{Prob(Up\ or\ Down) = \frac{40}{40} = 1 = 100\%}$$

So, probability has to be between 0% and 100%

$$\mathbf{0 \le \mathit{Prob}(\mathit{event}) \le 1}$$

[Note: As soon as we start talking about groups of outcomes we are really talking about ‘**events**’. An **event** is a **set** of outcomes or a particular situation that can arise during an experiment trial. Drawing a heart from a deck of cards would be an outcome, drawing a heart or a club would be an event. We can generally use the two terms interchangeably].

9. The probability that an event (or outcome) **does not** occur is called the **complement** of a probability, and is written as either ‘**~E**’ or $\bar{\boldsymbol{E}}$ where **E** refers to the **event**. In this case, E refers to the spoon coming to rest in the '**up**' position. Therefore, the probability that the spoon would **not** land '**up**' is:

$$\boldsymbol{P(\bar{E}) = 1 - P(E)}$$

The probability of something NOT happening is 100% minus the probability it did happen

or with a bit of algebra (juggling) we could say:

$$\boldsymbol{P(E) = 1 - P(\bar{E})}$$

So $P(Spoon\ NOT\ UP) = 1 - \frac{13}{20} = \frac{7}{20} = 0.35 = 35\%$

10. **Example 1**: If there is a 90% probability that you **will** succeed in passing Grade 12, what is the probability you will ***not*** succeed?

$$\boldsymbol{P(not\ succeed) = 100\% - 90\% = 10\%}$$

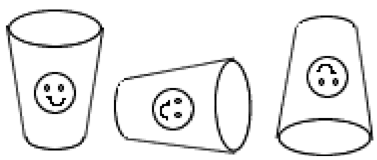
11. **Example 2**: If there is a 3 out of 5 chance of **no rain** today, what is the chance it **will** rain?

$$\boldsymbol{P(Rain) = 1 - P(\overline{Rain}) = 1 - \frac{3}{5} = \frac{2}{5} = 40\%}$$

1.3. Experimental Probability

12. Sometimes the calculation of probability is based on experimental data or on observations and experiments and trials over a period of time as we have discussed. For example, the probability of birds migrating on a particular date can be based on the observed migration dates of previous years.
Probabilities determined in this way are **Experimental Probabilities**.

13. Do the following experiment:
Drop a paper or foam cup to the floor 50 times, and record the number of times it comes to rest on its base, its side, and its top. A table similar to the one shown may be used to collect data.



Position	Tally (count)	No. of Trials	Prob
base			
side			
top			
Total		50	

14. Determine the **experimental** probabilities of the cup landing on its base, side and on its top. Write your answers in probability form rounded to two decimal places. Compare it with classmates. Discuss.

Note: The probabilities calculated in this situation are experimental because the calculation of the probability of each event is based on experimental observations (data). Sometimes called empirical probability.

1.4. Theoretical Probability

15. A bag contains **12** marbles (5 **red**, 4 **blue**, and 3 **green**). What is the probability of randomly selecting a red marble if one marble is taken from the bag?

Note: "**randomly selecting**" means that any marble in the bag has an equal chance of being selected; there is no '*bias*' in the experiment (like one of the marbles is not glued to the bottom or made a rough surface so someone could cheat and bias the experiment!)

16. We can determine the theoretical probability by using the following same probability formula as before:

$$\boldsymbol{P(Red)} = \frac{\boldsymbol{n(Red\,marbles)}}{\boldsymbol{n(Total\,marbles)}} = \frac{\mathbf{5}}{\mathbf{12}} \cong \mathbf{41.7}\%$$

Theoretically we would select a red marble on average five times every 12 tries (or about 42% of the time) if we repeated this activity many times.

The difference with this theoretical and experimental probability was that we did not actually have to do the draws, since each of the 12 marbles was equally likely to be drawn, we could just use the arithmetic.

The formula for theoretical probability is the same as for experimental

$$\boldsymbol{Prob_{Theoretical}} = \frac{\boldsymbol{n(favourable\ outcomes)}}{\boldsymbol{n(total\ possible\ outcomes)}}$$

However, when doing the counting, each outcome has to be equally likely.

Difference Between Experimental and Theoretical Probability

Theoretical probability is what we **expect to happen**, whereas experimental probability (sometimes called empirical probability) is what **actually happens** when we try it out. The probability is still calculated the same way, using the number of possible ways an outcome can occur as a ratio with the total number of outcomes.

Think About It! If the experimental result of an experiment and the theoretical expectation of an experiment are considerably different then either you did not do enough experimental trials or your theory needs to be modified. You have likely observed this often! Discuss.

1.5 Using Simulation to Determine Experimental Probability

17. Say you do a survey of all local families that have three children. What is the probability that there will be exactly two girls in a *family* with three children? We could wander about the neighbourhood and do a statistical survey of hundreds of three child families or we could experimentally '*simulate*' we wandered about the neighbourhood. To determine this experimentally with simulation, we can use three coins (three pennies) to represent the three children, with heads representing the girls.

(btw: Probability and Statistics are very closely related fields of study)

Toss the three coins and record the results. Two heads and one tail in any order would represent a family with two girls and one boy. Repeat your coin tosses until you have 50 trials. A table like the one shown could be used to record the data.

2H, 1T **ie: 2 girls,1 boy**	**Not (2H, 1T)** **ie: Not (2 girls, 1 boy)**

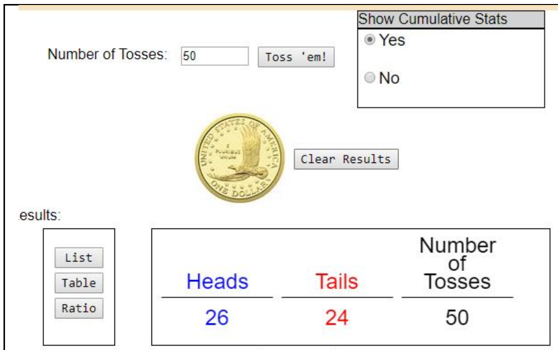
18. Determine the *experimental probability* of having two girls and one boy, the order does not matter, in a family with three children by simulation.

$$P(2G, 1B) = P(2Heads, 1Tail) = \frac{\#\ of\ trials\ with\ 2H\ and\ 1\ Tail}{Total\ \#\ of\ trials}$$

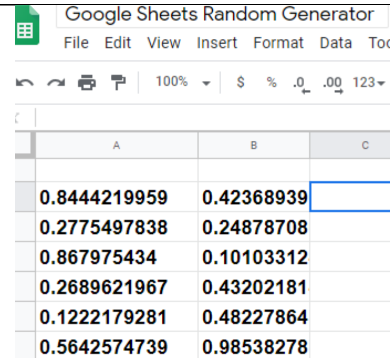
19. Later in this module, when we learn how to count, we will determine the theoretical probability of a family with three children having two girls and one boy. The **theoretical probability** for this kind of family is actually exactly **0.375**. Compare your answer to the theoretical value. Do you think that your answer would improve if you used more simulation trials? Discuss.

1.6 Using Technology to Simulate Probability Problems

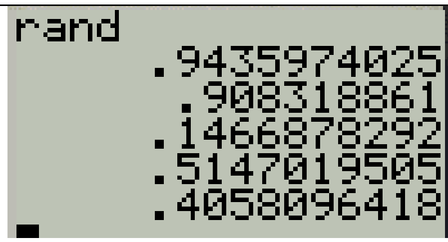
20. The TI-83 Graphing Calculator and Microsoft EXCEL spreadsheets and Google Sheets and dozens of websites are effective and fun ways to *simulate* some probability problems. Even a Dollarama Calculator.



http://www.shodor.org/interactivate/activities/Coin/



You know how to generate lots of random trials with your phone?



The random function on any calculator!

Simulations on the TI-83

Simulation of War

A special appreciation of war can be done by simulating you are in "**Number 6 Royal Canadian Air Force Bomber Group**" in England during WWII. Use the TI-83 Calculator, or any other random number generator device. If you generate a **0 to 0.970** then you **survived** one bombing mission (trial). If you 'generate' a **0.971 to 0.999** then you were killed on that bombing mission (trial). Even your Dollarama calculator will do these random numbers!

Can you do a tour of duty consisting of 30 bombing missions without dying?? Try it! Try it a few times for more tours of duty to see what percentage of the time you survived your 30 missions tour of duty.

What is the probability you returned to Canada after your mandatory 30 bombing missions if you were flying in Number 6 Royal Canadian Air Force Bomber Group? Simulate it on a device of some sort.

If you had 50 good buddies there, how many would you *expect* to survive?

(Answers: You likely did not return home about 60% of the time, likely about 30 out of your 50 buddies died, 20 lived; your experiment(s) should have given somewhere *around* those numbers unless you were very lucky, or very unlucky!)

EXPECTED VALUE

The probability formula can be **worked backwards** of course to predict how many actual times something is *likely* to happen if we know the probability. If we know there is a solid 50% chance of flipping heads on a coin, then after 30 flips we can expect to have 15 heads.

If **P(E)** = $\frac{n(E)}{n(total)}$ then: $\boldsymbol{n(E) = n(total) * P(E)}$

So, n(Heads) = Total flips * 0.5 = 30 * 0.5 = 15 heads

So, we can **expect** to get 15 Heads. On average; most of the time! Maybe 13 or 14, maybe 16 or 17, but on average we can **EXPECT** to get 15 Heads.

Hence, we call this **EXPECTED VALUE**.

Simulate it; see what happens! (unless you feel like actually flipping real coins and recording results a couple thousand times!!)

Activity from: http://www.shodor.org/interactivate/activities/Coin/

Clear Results

Heads	Tails	Number of Tosses
14	16	30

Clear Results

Heads	Tails	Number of Tosses
15	15	30

Clear Results

Heads	Tails	Number of Tosses
9	21	30

Wow! Someone was lucky (or unlucky) this simulation!

Clear Results

Heads	Tails	Number of Tosses
13	17	30

You use expected value many times a day!! Discuss.

Ever watch PLINKO on the Price is Right? How much money can you *expect* to win with four discs? Ask teacher to calculate! Fun!

Example (Gambling!): If you play a VLT 10,000 times how many times can you expect **any four of a Kind**?

The exact theoretical probability of getting any four of a kind is exactly

$$\frac{\mathbf{624}\ \boldsymbol{Wins}}{\mathbf{2{,}598{,}960}\ \boldsymbol{Plays}}$$

(you will learn how to calculate this when we learn how to count later in these notes!)

So $\boldsymbol{n}(\boldsymbol{E}) = \boldsymbol{n}(\boldsymbol{total}) * \boldsymbol{P}(\boldsymbol{E})$

Gives n(4 of a kind) = 10,000 * $\frac{624}{2{,}598{,}960}$ = 2.4 times on average every 10,000 plays you will get some four of a kind.

So, you can spend $10,000 to likely win $2,400. 👎

Adding in other smaller wins does not help much either.

But maybe you are super lucky and win 3 times in 10,000 plays!

Or maybe you win hundreds of times! LOL; not in the history of the universe would that ever happen!

ODDS (Another Measure of Chance)

There is more than one way to express the chance of something happening or not happening!

Probability = $\frac{\#of\ favourable\, outcomes}{\#\ of\ total\ possible\, outcomes}$

Odds is basically the same but instead of comparing **what you want** with what **you could have**, you are comparing **what you want** to what you **don't want**! Just that easy!

There are **two forms** of odds:

Odds in Favour of something happening;

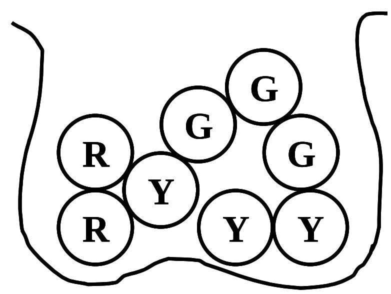
Odds in favour = **# of favourable outcomes : # of *un*favourable outcomes**

Odds Against something happening;

Odds against = *# of **un**favourable outcomes* **:** *# of favourable outcomes*

Notice we tend to not write odds as a fraction but as a ratio with a colon symbol, **:**, especially to avoid confusion with the probability ratio. Or some time just the word '**to**'.

Example: If the favourable event is drawing a Green marble from the bag then:



Prob (Green) = $\frac{3}{8}$ or 0.375 or 37.5%

Odds in Favour of Green = **3 : 5**

Odds Against Green = **5 : 3**

1.7 Lesson Summary

21. The following words (terms) were used in this lesson:

- complement of a probability
- event
- experimental probability
- outcome
- probability
- random selection
- simulation
- theoretical probability
- expected value
- odds

You should know how to use the following formulas to solve problems covered in this lesson. Make sure you have them on your Study Notes (cheat sheet)

$$\boldsymbol{Prob = \frac{n(favourable\ outcomes)}{n(total\ possible\ outcomes)}}$$

$$\boldsymbol{P(\bar{E}) = 1 - P(E)} \quad ; \quad \boldsymbol{P(E) = 1 - P(\bar{E})}$$

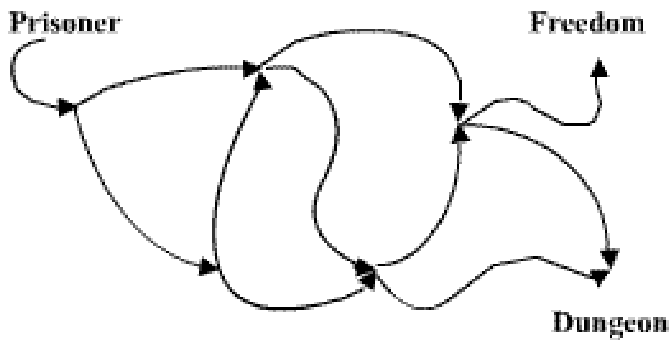
Expected Value (EV): $\boldsymbol{n(E) = n(total) * P(E)}$

Lesson 2 - Solving Pathway Problems Using Diagrams

Learning to Count!

2.1. Introduction

22. An old legend describes a situation where a prisoner must walk through a maze of branching tunnels that wind up in one of two places - freedom or a dungeon. If the prisoner is equally likely to turn left or right at any branch in the path, how likely is he to gain his freedom? His only constraint is that he cannot backtrack. He must continue in the directions indicated by the arrows.



Did you get the answer?

In this lesson, we will solve problems like the one above by drawing pathway diagrams and calculating the theoretical probability.

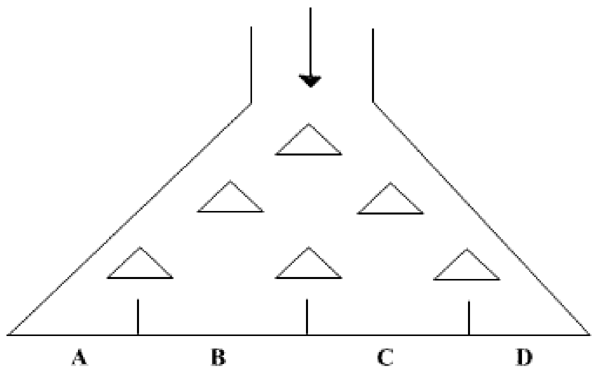
2.2. Outcomes

When you have completed this lesson, you will be able to:

- determine the number of pathways through a pathway map
- determine the probability of selecting any particular pathway

2.3. A Pinball Game (PLINKO!)

23. The diagram below shows a game of chance where a ball is dropped as indicated, and eventually comes to rest in one of the four locations labelled **A, B, C**, or **D**. The ball is *equally likely* to go **left** or **right** each time it strikes a triangle. We want to determine the *theoretical probability* of a ball landing in any one of these four locations. To do this, we need to count the total number of paths the ball can take to the bottom, and also count the number of paths to each individual outcome bin at the bottom.



24. The ball may take the following routes through the game.

Bin	Possible Paths (set of steps)	Count
A	{Left, Left, Left}	1 way
B	{Left, Right, Left}; {Left, Left, Right}; {Right, Left, Left}	3 ways
C	{Left, Right, Right}; {Right, Left, Right} {Right, Right, Left}	3 ways
D	{Right, Right, Right}	1 way

The total number of routes through the game is

$$1 + 3 + 3 + 1 = 8$$

The probability of coming to rest in bin A could be determined as follows:

$$P(Bin\ A) = \frac{n(routes\ to\ A)}{n(total\ routes)} = \frac{1}{8} = 12.5\%$$

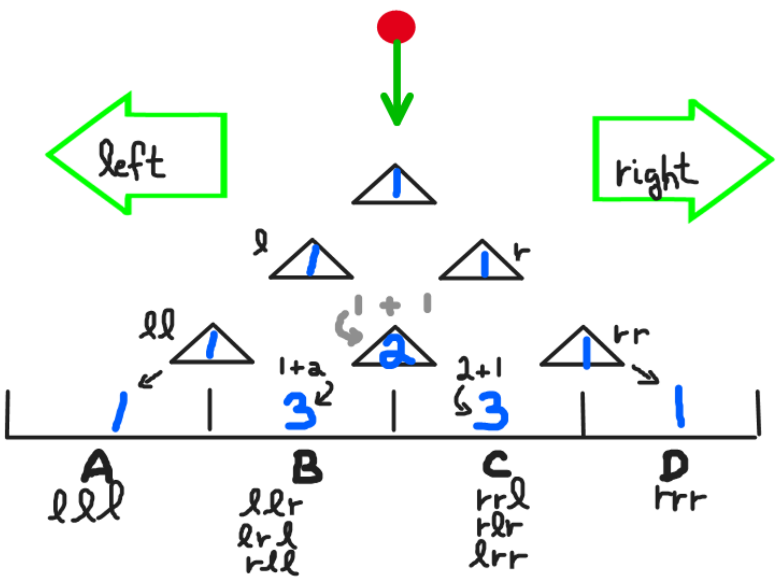
Similarly,

$$P(Bin\ B) = \frac{n(routes\ to\ A)}{n(total\ routes)} = \frac{3}{8} = 37.5\%$$

etc.

3.3.2. Another Look at the Pinball Game (Pascal Triangle)

25. The number of ways that the ball can proceed through the game may also be indicated with numbers, as shown in the diagram below. Each number indicates the number of routes possible for the ball at any particular location.



Note that each pathway to a location is the sum of the previous pathways either side and above each decision 'node'.

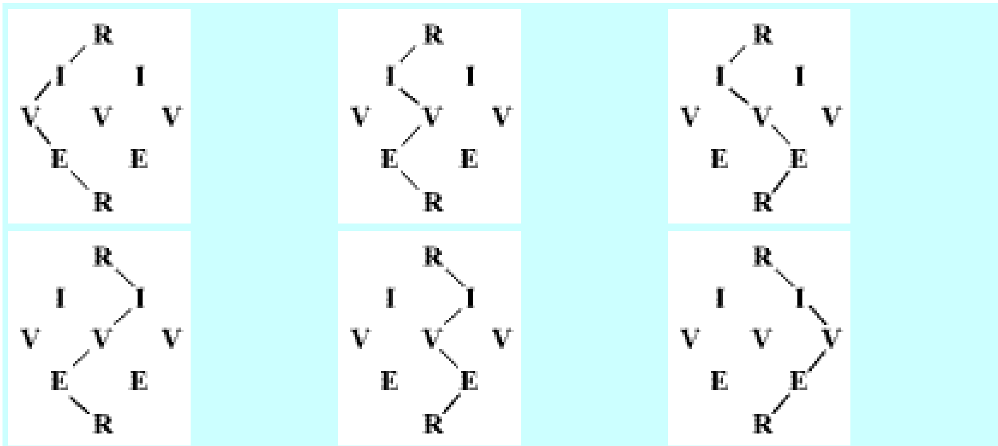
If you want to research this triangle of pathways more checkout the mathematician '**Pascal**'.

2.4. How Many Rivers?

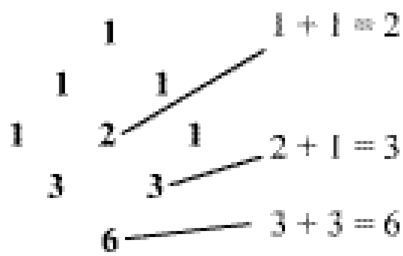
26. How many ways can the word **RIVER** be found in the array of letters shown to the right if you start from the top R and move down to the bottom R?

R
I I
V V V
E E
R

The diagrams below show six solutions.



Note that the number of routes up to any letter is the sum of the routes through previous letters above, as shown in the diagram to the right.

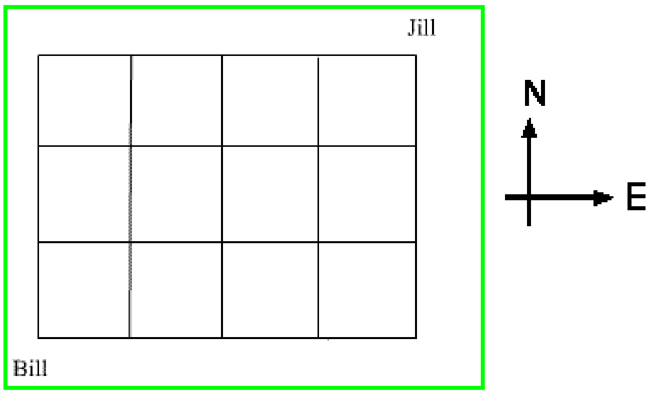


27. Determine the probability that the pathway goes through the middle V

$$\boldsymbol{P(bounce\ through\ middle\ V) = \frac{n(paths\ through\ middle\ V)}{n(total\ paths)} = \frac{4}{6} = \frac{2}{3}}$$

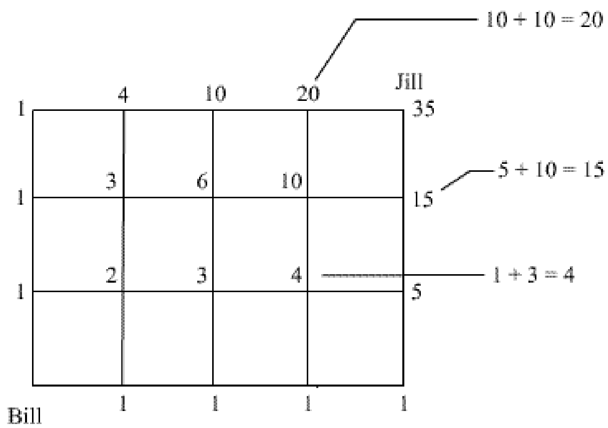
2.5. How Many Ways Can Bill Go to See Jill?

28. Bill lives seven blocks from Jill, as shown on the diagram. He has to move **7 moves**; eventually some combination of **4 moves** have to be **east** and **3 moves** have to be **north**. How many different routes can he take to her place if he walks only **east** *or* **north**? (Not staggering around!)



2.6. Bill's Answer

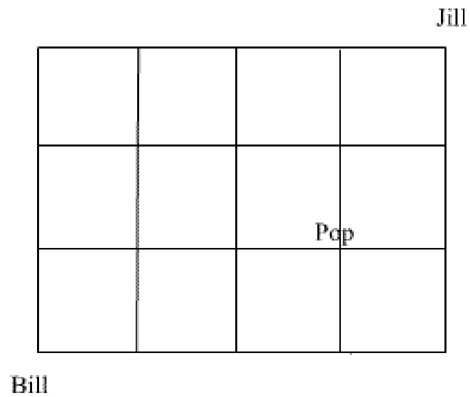
29. The diagram shows the number of pathways at each intersection. The number of pathways at any intersection is the sum of the pathways at the previous intersections, as shown.



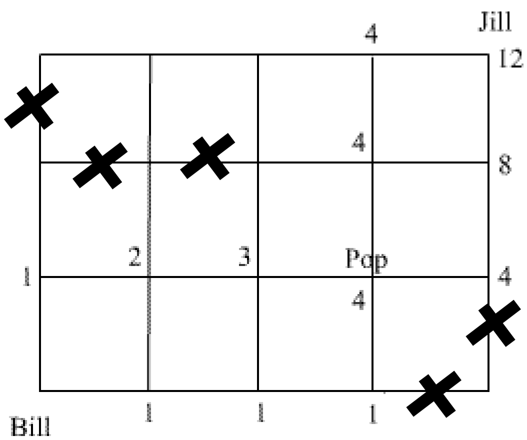
Therefore, the total number of routes to Jill's house is 35.

2.7. What Is the Probability of Passing the Pop Shop?

30. The Pop Shop in town is located on a street corner as shown on the diagram. What is the probability that Bill will pass the Pop Shop on his way to Jill's house if his route is selected randomly?



31. The numbers at the intersections indicate the number of routes that include the intersection where the Pop Shop is located. This is a constraint or restriction type of question because we are only interested in counting the pathways that go past the pop shop.



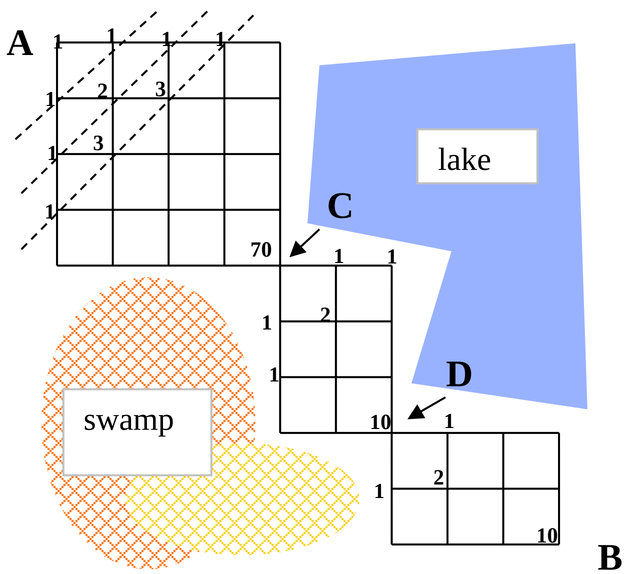
Therefore, the probability of passing the Pop Shop is calculated as follows:

$$Prob(walk\ past\ Pop\ Shop) = \frac{n(routes\ past\ pop\ shop)}{n(total\ routes)} = \frac{12}{35} \cong 34.29\%$$

2.8. Pathways with Constraints (Advanced)

32. This is an advanced section which will be done after learning the Fundamental Counting Principle.

You may be aware that the entire rural prairies of Canada are set up in a grid of roads and section lines, all exactly one mile apart, from Manitoba to the Rockies. How many pathways are there from Adele's farm, **A**, to Barry's farm, **B**. Given that the lake and the swamp constrains the paths. You may only move south or east, you are not allowed to manoeuvre away from your destination.



33. The number of possible paths from **A to B** is *constrained* to pass through **C and D**. **C and D** are like 'pinch' points. So, all that needs be done is calculate the number of paths from **A to C**, **C to D**, and **D to B** and then multiply them. We multiply of course because of the FCP (Fundamental Counting Principal).

34. **A to C** is 70, **C to D** is 10, and **D to B** is 10. The Total number of possible paths is therefore **7000**.

btw: there is a button on your calculator does these in about 20 seconds! Wait for it

2.9. Lesson Summary

The following words (terms) were used in this lesson:

Constraints & restrictions
Pathways
Pascal Triangle

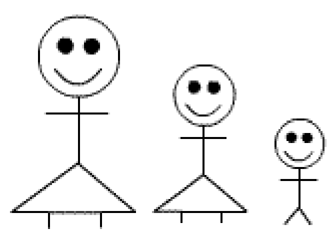
Although there were no new formulas introduced in this lesson, you learned the following:

- how to determine the number of pathways through a pathway map
- how to determine the probability of arriving at a particular location when given certain constraints
- (*Advanced*) If you want to know more about these types of problems investigate the 'PASCAL' triangle!
- We will also discuss these pathways again, solving them with just math, when we discuss Permutations and Combinations and the Fundamental Counting Principle.

3. Lesson 3 - Using Sample Spaces to Solve Probability Problems

3.1. Introduction

35. How many daughter - son arrangements are possible in a three-child family? What is the probability of having two daughters followed by one son? To answer these questions and others like them, we need to know the total number of possible outcomes.



A **sample space** is a list of all possible outcomes. In this lesson, we will write and use sample spaces to solve probability problems.

3.2. Outcomes

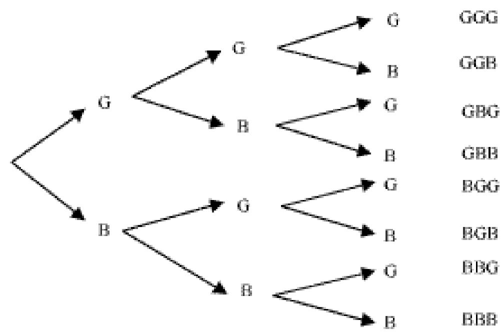
When you have completed this lesson, you will be able to:

- generate a sample space using a tree diagram
- use sample spaces to solve probability problems

3.3 Tree Diagram of Three Children

36. A **tree diagram** or **outcome tree** is a branching diagram that shows all the possible outcomes of an experiment.

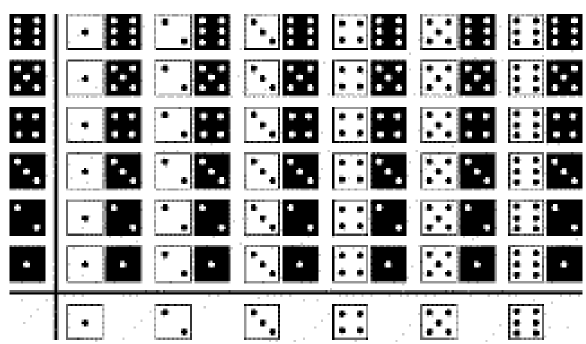
Draw a tree diagram to show all possible boy-girl outcomes in a family if there are three children.



The eight possible outcomes are listed to the right of the tree diagram. This list of eight outcomes is the **sample space**.

3.4. Write a Sample Space Using a Table

37. A certain experiment consists of many trials of rolling two regular dice and recording the **sum** of the two dice. We are measuring a **compound event** here, two simple outcomes of two dice being **combined** to make a **sum**. The sample space outcomes are illustrated below.



38. The sample space may be written in table form like the one shown below. This is a convenient way to record all the outcomes of the sample space of the sum of two dice. **Each of these outcomes is equally likely**.

SUM	First Die					
Second Die	2	3	4	5	6	7
	3	4	5	6	7	8
	4	5	6	7	8	9
	5	6	7	8	9	10
	6	7	8	9	10	11
	7	8	9	10	11	12

39. The numbers on the table represent the **sum** of two dice.

Determine the following probabilities by counting. Reduce all fractions to lowest terms, and enter your answers in the boxes provided.

What is the probability of a sum of 7 being rolled?	
What is the probability of a sum of 12 being rolled?	
What is the probability of a sum of one being rolled?	
What is the probability of the *event* of a sum greater that 8 being rolled?	

*An **event** is a set or combination of outcomes*

3.5. Lesson Summary

40. The following words (terms) were used in this lesson:

- sample space
- outcomes
- tree diagram (outcome tree)
- event

You should know how to do the following operations:

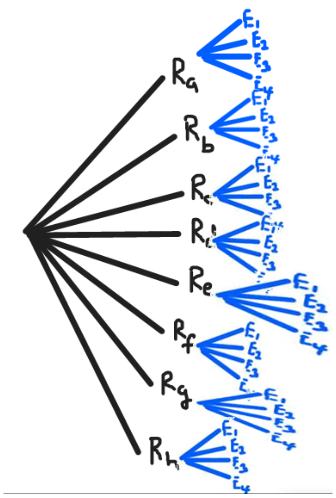
- draw a tree diagram showing all the outcomes of two or more events;
- write a sample space showing all the outcomes; and
- write a sample space using a table or chart.

4. Lesson 4 - The Fundamental Counting Principle (FCP)

4.1. Introduction

41. A group of students on a tour are planning the evening's activities together. They have a ***choice*** of **eight** restaurants (**R_a, R_b, R_c,** …) and *then* a **choice of four possibilities** for entertainment (**E_1, E_2, E_3, E_4**). How many different restaurant and entertainment **arrangements** are possible?

42. We could draw a tree diagram or write a sample space to describe the above situation and then count the number of branches or the elements in the sample space, but this would be difficult because there are too many outcomes. In this lesson, we will learn how to determine the total number of outcomes without actually counting all of them individually.



4.2. Outcomes

When you have completed this lesson, you will be able to:

- use the Fundamental Counting Principle (FCP)
- solve probability problems ***without*** using sample spaces or tree diagrams

4.3. How Many Different Bicycles?

43. A certain brand of bicycle is available in three colours (yellow, blue, and red), two tire types (smooth, grip), and two different seats (economy, custom). How many different bicycles could we chose to buy from?

44. We could determine the answer by drawing a tree diagram or writing the sample space and counting the outcomes. The sample space is shown below. The first letter refers to the colour, the second letter to the tire type, and the third letter to the type of seat. The three letters RGC would indicate the outcome Red colour - Grip tire - Custom seat.

{YSE}	{BSE}	{RSE}
{YSC}	{BSC}	{RSC}
{YGE}	{BGE}	{RGE}
{YGC}	{BGC}	{RGC}

4.4. The Fundamental Counting Principle

47. When two or more choices must be made together, the total number of outcomes can be determined without listing and counting them. The rule for this is known as "**The Fundamental Counting Principle**."

The **Fundamental Counting Principle** states the following:

"*If one event can occur in '**a**' ways, a second event in '**b**' ways, a third event in '**c**' ways, and so on, then the number of ways that all events can occur one after the other is the product **a * b * c. . .** .*"

4.5. Example 1: Fundamental Counting Principle

48. Sue has four pairs of shoes, five pairs of jeans, and seven sweaters. How many different clothing combinations can she select?

A useful technique to use when solving this question is to draw spaces to represent the number of events, and then writing the number of ways each event can occur. The three events are shoes, jeans, and sweater:

$$\frac{\quad\quad}{\text{shoes}} \text{ x } \frac{\quad\quad}{\text{jeans}} \text{ x } \frac{\quad\quad}{\text{sweater}}$$

and the number of ways each choice can be made is written in the space:

$$\frac{4}{\text{shoes}} \text{ x } \frac{5}{\text{jeans}} \text{ x } \frac{7}{\text{sweater}}$$

49. The total number of clothing combinations possible is:

$$\underline{4} \times \underline{5} \times \underline{7} = 140$$

Is it surprising that getting dressed in the morning can be such a challenge for Sue if she has **140** different clothing combinations to choose from?

4.6. Example 2: Fundamental Counting Principle

50. Manitoba license plates are formed by using three letters of the alphabet and three numbers in the form: ABC 123. (there are 26 letters in the English alphabet)

We need to select three numbers and three letters and arrange them as follows:

_______	_______	_______	_______	_______	_______
Letter	Letter	Letter	Number	Number	Number

51. How many choices do we have for the first character of the License Plate? 26
How many for the second character? 26. Etc. You see the number of choices for each character on the license plate is therefore:

__26__	__26__	__26__	__10__	__10__	__10__
Letter	Letter	Letter	Number	Number	Number

Remember the **FCP** (**Fundamental Counting Principle**!)

If one event can occur in 'a' ways, a second event in 'b' ways, a third event in 'c' ways, and so on, then the number of ways that all events can occur one after the other is the product a*b*c. .

52. Therefore, the number of each of the six events of selecting a character for the positions on our license plate are can be multiplied together:

So: 26*26*26*10*10*10 = 17,576,000 ways

It will be many centuries before Manitoba uses all the possible license plates!

4.7. Factorial Notation

53. How many ways can the letters of the word **MONKEY** be arranged?

Solution:

There are six letters to choose (and, therefore, six spaces in which to place them) and we make the choices as indicated.

Therefore, there are:

6*5*4*3*2*1 = 720 arrangements

Note: When we want to multiply all the natural numbers from a particular number down to 1, we can use **factorial notation** to indicate this operation. The symbol **"!"** is used to indicate factorial. This notation can save us the trouble of writing a long list of numbers.

54. There is nothing magic about factorials, they are not divine gifts from some creator, they are just a simple way that mathematicians use to show they multiply together a bunch of whole numbers down to one.

For example:

6! means 6 * 5 * 4 * 3 * 2 * 1 = 720

4! = 4 * 3 * 2 * 1 = **24**

10! = 3,628,800

1! = 1 (one way to arrange one thing)

and we *define* 0! = 1
(There is one way to arrange zero things!)

55. Check your calculator to find the **"!"** notation. If you are using the TI-83, the notation is found at: **MATH > PRB > 4 .** Somewhere on your calculator will be a button like
x! or **n!**

You Try. Evaluate the following factorials ***without*** using a calculator
Show work!

3! = $\mathbf{3 \cdot 2 \cdot 1 = 6}$	5! =	10! =	49! =

You can see that 49! would take a very long time to calculate without a decent calculator and likely calculate with lots of errors regardless.

56. Calculate the following **with a calculator** now:

8! =	12! =	13! =
32! =	0! =	3.5! =

Notice that factorials only work for the set of whole numbers. Fractions and negative numbers do not work obviously because we are just counting things. The TI-83 will give an answer to rational number factorials, even though they are not legal.

Notice also that 13! is just 13 times 12 factorial! i.e.: **13! = 13*12!**

4.8. Example 3: Fundamental Counting Principle Using the Permutation Formula

We introduce the term: ***Permutation***	The **number** of ways of **selecting** and **ordering** objects is a permutation (two actions).

57. How many different five-letter "*words*", even if they are nonsense arrangements, can be made from the [English] alphabet if no letters are repeated, that is no letter is re-used once we select it? A "word" in this case is any five-letter arrangement with all letters different. When we start to put special condition on what cannot be done in making our arrangements of items, we are dealing with permutations that have *restrictions*.

Solution:

We will use the expression

$${}_n\mathbf{P}_r = \frac{\boldsymbol{n}!}{(\boldsymbol{n}-\boldsymbol{r})!}$$

to calculate the number of ways to make an ordered arrangement of '**r**' number of things selected from '**n**' number of things.

to answer this question. The value of '**n**' is 26 (there are 26 objects from which to choose in English) and the value of '**r**' is 5 (we pick five different letters).

$${}_nP_r = \frac{26!}{(26-5)!} = \frac{26*25*24*23*22*21!}{21!} = 7{,}893{,}600 \text{ words}$$

58. The permutation formula may be convenient to use, but it is not essential. You can always find the number of arrangements (permutations) by FCP multiplication as shown below.

26*25*24*23*22 = 7,893,600 words

Notice that the order of the items selected is important in a permutation. If all we had to do was grab 5 letters and stick them in our pocket that would be a slightly different calculation.

59. **The Permutation function**. The function ${}_nP_r$ is defined as:

$${}_nP_r = \frac{n!}{(n-r)!}$$

for the number of ways to select ***r*** objects at a time without replacement from ***n*** different objects and placing them in an ordered arrangement. (Notice how ***n*** has to be greater than or equal to ***r****, you cannot select more thangs than you have*)

Caution: some text books and calculators use the notation: **P(n,r)** instead of **${}_nP_r$**; they mean the same thing and describe the same formula.

Check out some of the permutation tables at the end of these notes. Tables of values can show neat patterns that a calculator never can.

60. **Expanding $_nP_r$.** Just so you understand what $_nP_r$ means, we will 'expand' and 'evaluate' a few examples. You fill in the rest of the table.

$_nP_r$	Expanded	Evaluated
a. $_5P_2$ How may ways can we select 2 objects from 5 different objects and place them in an order?	$\frac{5!}{(5-2)!}$ $\frac{5!}{3!}$ $\frac{5*4*\cancel{3*2*1}}{\cancel{3*2*1}} = 5*4 = 20$	=20
b. $_8P_5$		
c. **P(7, 2)**		
d. $_{48}P_{24}$		
You should see from **d.** above why mathematicians wanted a better way to write permutations! Saves a lot of ink using the Permutation Function and Factorials		

61. **More properties of the Permutation Function; $_nP_r$.**

Remember that **0!** is **defined** as **1**. There is one way to arrange no things.

So, what is $_5P_5$?, in other words how many ways can you select 5 objects without replacement from 5 different objects and place them in an order?

In notation this looks like: $_nP_r = \frac{5!}{(5-5)!} = \frac{5!}{0!} = 120$

So that is one reason why we had to define **0!** to be equal to 1. Otherwise we would have needed a more complicated formula.

62. You can find the permutation function on A TI-83 Graphing Calculator. It is at **MATH PRB 2:nPr.** To use the function on the TI-83, enter the ***n*** on your screen, then select the $_nP_r$ function, select ENTER , then enter the ***r*** on your screen. The press ENTER to do the calculation. Find the $_nP_r$ button on your own calculator, any decent calculator will have it.

Evaluate the following using a TI-83 or similar calculator

a. $_5P_5$	b. $_7P_2$	c. $_4P_2$
d. $_{25}P_1$	e. $_{25}P_4$	f. $_{25}P_{25}$

Did you get something wild like 1.551121**E**25 for **f.** above? This is how calculators and computers show powers of ten. 1.551121E25 really means $1.551121*10^{25}$ or in other words:

15,511,510,000,000,000,000,000,000

which is **1.551121** and then moving the decimal point 25 places to the right. This relates to '*Scientific Notation*' from earlier Mathematics. It is such a huge number that the calculator cannot show it all so it shows you a decimal number as best it can and then shows how many places to move the decimal!

4.9. Permutations with Non-Distinguishable Objects

63. Sometimes not all the objects from which we have to select are distinct (or '*distinguishable*'). For example say we want to see how many ways we can pick distinct words from arrangements of **all** the letters in the word **RIVER**. ***If*** the letters were all different it would just be **5!** or **120** ways. But notice there are **two Rs**.

The formula we need to use is $\frac{5!}{2!}$ or **60** because there are two **R**s

RULE: The number of distinguishable permutations of **n** objects of which there are **a** objects alike of one kind, **b alike** of another kind, **c alike** of another kind, and so on, is $\frac{\boldsymbol{n}!}{\boldsymbol{a}!*\boldsymbol{b}!*\boldsymbol{c}!\ldots}$

Example: How many possible ordered arrangements using all the letters in the word **PINGPONG** are there?

Answer: $\frac{\mathbf{8}!}{(\mathbf{2}!*\mathbf{2}!*\mathbf{2}!)\ldots} = \mathbf{5040}$; since there are **2 P**s, **2 N**s, and **2 G**s.
Do not forget the entire 2!2!2! expression in the denominator has to be in brackets in your calculator

64. **You try!** Flora is off to **Bingo**. She has **three Blue** Bingo dabbers, **two Red** Bingo dabbers, **two Pink** Bingo dabbers, and **one Green** one. How many ways can she make distinct arrangements of her bingo dabbers in a line in front of her. (they are all exactly the same brand, the only thing that makes them distinguishable is their colour). Show your formula and calculation below. (*Ans: 1680*)

65. If she loans **one** of her **Blue** Bingo dabbers to a friend, how many ways can Flora now distinguishably arrange her Bingo Dabbers now in front of herself in a line? (*Ans: 630*)

4.10. Introduction to Combinations

66. In the previous examples, the **order** in which events occurred **was important**. We now want to look at counting problems where the **order does not matter**. This type of arrangement, a **combination**, is often much more important in everyday life and probability questions.

67. **Question**. A committee of three is formed from five students in a class. How many ways can this be done?

Solution:

68. The order is not important because the committee members of **Albert**, **Bruce**, and **Carol** is the same as the committee of **Albert**, **Carol**, and **Bruce**. This type of problem, where a set of objects is selected with no regard to the order, is called a **combination**. Arrangements ABC, and ACB, and BCA, etc, (All 6 of them) are non-distinguishable if order does not matter.

We begin the calculation the same way as a permutation.

$5 * 4 * 3 \; or \frac{5!}{2!}$ = 60

69. Now we must divide by the number of ways in which the three non-distinguishable committee positions can be arranged among themselves. There are 3 * 2* 1 = 6 ways to juggle around the positions if the order does not matter

60 / 6 = 10 arrangements of committee if there is no particular order.

The factorial method is very convenient if we are dealing with large numbers and if we are using a scientific calculator.

4.11. Example 1: Combinations

Answer the question before looking at the solution.

70. There are 12 members on a First Nations' school track team. Only three of them can be selected to participate in the International Aboriginal Games in New Zealand. What is the total possible number of three-person teams that are possible?

Number = $\frac{12!}{(12-3)!\,3!} = \frac{12!}{9!\;3!}$ = 220 possible ways to choose a three-person team for the International Aboriginal Games

4.12. Example 2: The Combinations Formula

71. How many ways can a committee of five be chosen from a group of 20 people?

The order in which the committee members are chosen is not important, and so this is a combination problem. The number of committees can be determined as follows:

$$\frac{20 * 19 * 18 * 17 * 16\ permutaions}{5 * 4 * 3 * 2 * 1\ ways\ to\ juggle\ the\ places\ of\ the\ permutation} =$$

$$= \frac{{}_nP_r}{r!}$$

15,504 un-ordered arrangements of the different members

Or you can think of it as two possible outcomes. 'Chosen, not chosen'. So, pretend you have 20 Bingo dabbers, 5 of one colour (chosen), 15 of the other (not chosen). The distinguishable arrangements are: ${}_nC_r$

$\frac{20!}{5! * 15!} = \frac{20!\ ordered\ people\ arrangements}{5!\ winners * 15!\ losers}$ = 15,504

72. The following formula may be used to determine the number of combinations if you use a scientific or graphing calculator. The 'n' refers to the number of objects from which you choose (in this case the number of people) and 'r' refers to the number of objects that are selected at a time (the number of committee members).

$$_nC_r = \frac{n!}{(n-r)! * r!}$$

Expanding and evaluating we get:

$$_{20}C_5 = \frac{20!}{(20-5)! * 5!} = \frac{20!}{(15! * 5!)} = 15{,}504$$

Check your calculator to find the combination formula. If you are using the TI-83, the notation is found at:

MATH > PRB > 3:nCr; any decent calculator for $12 will have it too!

73. You Try. **Expand** and **Evaluate** *without a calculator* the following:

$_4C_1$	$_6C_2$	$_6C_4$

74. Evaluate with a calculator now: (btw: several of these can be done in your head or with a Pascal Triangle!)

${}_{49}C_6 =$	${}_{20}C_5$	${}_{7}C_4 =$
${}_{13}C_6 =$	${}_{13}C_7 =$	${}_{20}C_0 =$
${}_{20}C_1 =$	${}_{20}C_2 =$	${}_{20}C_{18} =$

Note: in some books you will see the Combination notation given as $\binom{n}{r}$ or as **C(n, r).** The formula to calculate them is still the same.

4.13. Example 3: Combinations

[A Calculator that can do the Combination Formula is highly desirable for this unless you want to do the evaluations the long and expanded way!]

75. A standard deck of 52 playing cards consists of 4 suits (spades ♠, hearts ♥, diamonds ♦, and clubs ♣) of 13 cards each.

a) How many different 5-card hands (poker) can be formed? (order doesn't matter when you are dealt a hand of cards)

$$_{52}C_5 = \frac{52!}{(52-5)!5!} = \frac{52!}{47!\,5!} = 2{,}598{,}960 \text{ ways}$$

b) How many different 5-card hands can be formed that consist of all hearts?

$$_{13}C_5 = 1287$$

See why we need to learn to 'count' to do probability? The probability of being dealt a flush in hearts is:

$$P(FlushHearts) = \frac{1287}{2{,}598{,}960} = 0.0005 = 0.05\%$$

or like 5 times out of every 10,000 dealt hands.

c) How many different 5-card hands can be formed that consist of all face cards?

$$_{12}C_5 = 792$$

d) How many different 5-card hands can be formed that consist of 3 hearts and 2 spades?

$$(_{13}C_3)\,(_{13}C_2) = 286 * 78 = 22308$$

(Notice the use of the FCP above)

e) How many different 5-card hands can be formed that consist of exactly 3 hearts?

$({}_{13}C_3)\ ({}_{39}C_2) = 211926$

76. Can you see this last one? The first factor tells us how many ways we could choose the first three cards. But we *can't forget* for every way we do that there are still two other cards we need to draw from a choice of 39 *remaining cards*.

Lotto6/49 and Gambling

77. Have you ever done a '*voluntary tax*' at this: **Lotto 6/49**, or **Super7**, or at **casinos**??

In the Lotto 6/49 for example, there are **49** balls that bounce around in a big plexi-glass sphere. Six balls roll out. If **your** selected combination of the six numbers is the same (regardless of order) then you win! What is your **probability** of winning with ***your*** **one** selection of numbers?

$$P(6/49\ jackpot) = \frac{1}{{}_{49}C_6} = 7.15*10^{-8} = 0.00000715\%$$

so once every **$_{49}C_6$** (ie: 13, 983, 816) draws you might expect to *maybe*, on *average*, win the jackpot **ONCE** (or maybe share it with others). So if you played for **135,000 years** (twice a week) you could expect to *maybe* win the jackpot **once** on average, and hopefully not share it with someone who had the same combination as you!

78. It is no wonder society calls gambling a tax on the stupid! Of course, we are all stupid sometimes!

Curious!! *If the order of selected objects or numbers doesn't matter in a combination; then why do we call our locks 'combination' locks? Surely the order in which we dial our 'combo' is important! Why isn't it called a permutation lock?*

4.13. Lesson Summary

The following words (terms) were used in this lesson:

- combination
- event
- factorial notation
- Fundamental Counting Principle
- permutation

You should know how to use the following formulas or concepts to solve problems covered in this lesson.

- Fundamental Counting Principle
- ${}_nP_r = \frac{n!}{(n-r)!}$ Permutations, ordered arrangements of r objects selected from n objects
- Distinguishable Arrangements: $\frac{\boldsymbol{n}!}{\boldsymbol{a}!*\boldsymbol{b}!*\boldsymbol{c}!\ldots}$
- ${}_nC_r = \frac{n!}{(n-r)!*r!}$ Combinations, **un** - ordered arrangements of r objects selected from n objects

5. Lesson 5 - Independent and Dependent Events

5.1. Introduction

79. In the average Manitoba high school, 40% of all the students get at least one cold each year. The public health nurse of **'Tween Lakes Collegiate'** encouraged all the students to take a certain brand of vitamin tablet to help reduce the number of colds. This year, 60% of the students did take the vitamin tablet, and 15% of the students who took the vitamin tablets also caught a cold.

Did the vitamin treatment do any good? Was the probability of getting a cold reduced, increased, or not affected by the vitamins?

In this lesson, we will learn how to answer this and other similar questions. Do some events affect other events?

5.2. Outcomes

When you have completed this lesson, you will be able to:

- identify independent and dependent events
- use the multiplication rule for probability

5.3. Definition of Independent Events

80. **Event A** and then **event B** are **independent events** if the outcomes of event A in no way influence the outcomes of event B. The probability that all of a set of independent events will occur is the product of their separate probabilities. This may be written as the formula:

P(A and B) = P(A) * P(B) for the probability of both event A and event B happening.

You might also notation like this: P(A $\cap$ B) if we later study sets and logic

(*Note that the word '**and**' implies multiplication in many cases in mathematics especially when used in a **logic** or a **probability** sense*)

81. For example, if you are flipping a coin and rolling a six-sided die, the outcome from rolling the die is in no way affected by the outcome of flipping the coin. Let A and B represent the following events:

A. flipping a coin and getting heads.
B. rolling a die and getting a two.

Then the following are the probabilities of each outcome:

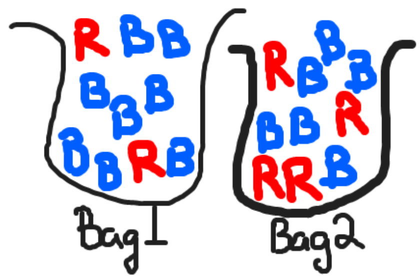
$P(A) = \frac{1}{2}$; $P(B) = \frac{1}{6}$; so P(A and B) = $\frac{1}{2} * \frac{1}{6} = \frac{1}{12} \approx 8.33\%$

Therefore, the probability of flipping both a head and rolling a two is 1/12.

5.4. Example 1: The Probability of Independent Events

82. Imagine you have two separate bags with coloured beads in them. You must choose **one bead from each bag**. The **first bag** contains two **red** beads and eight **blue** beads. The **second bag** contains four **red** beads and six **blue** beads. Picking from the first bag and picking from the second bag are ***independent*** events.

a) Find P(Red from the first bag) or
in shorter form: $P(R_1)$



b) Find P(Blue from the first bag)
i.e.: $P(B_1)$

c) Find P(Red from the second bag)
i.e.: $P(R_2)$

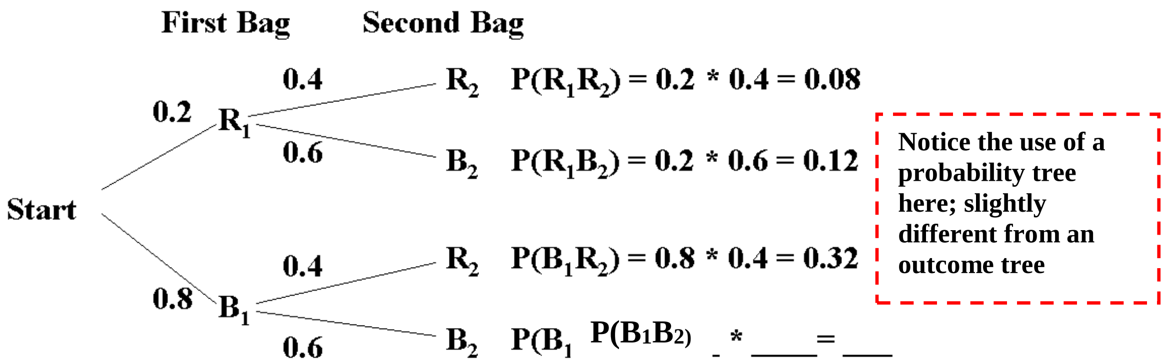
d) Find P(Blue from the second bag)
i.e.: $P(B_2)$

83. Now answer these questions.

a) What is the probability of getting two reds? $P(R_1 \text{ AND } R_2)$

b) What is the probability of getting one red and then one blue **or** one blue and then one red? $P((R_1 \text{ and } B_2) \text{ OR } (B_1 \text{ and } R_2))$

c) What is the probability of getting **at least** one red
ie: $P(\geq 1 \text{ Red}) = (P((R_1 \text{ AND } R_2) \text{ OR } (R_1 \text{ AND } B_2) \text{ OR } (B_1 \text{ AND } R_2))$?



84. The Prob(Two Red) is **0.08 or 8%.** We multiply the probability of the outcomes.

85. The Prob (Red then Blue **or** Blue then Red) is **0.44 or 44%.** (notice the event of RED and BLUE on either of two draws has two outcomes: (R_1,B_2 **or** B_1,R_2)

86. Prob (at least 1 red) can be written:

Prob(≥1RED) = Prob (1 RED) + Prob(2 RED)
=(0.12 + 0.32) + 0.08 = 0.52 or 52%

*Another way to solve this last question, "how many outcomes have **AT LEAST ONE RED**", is just to find out the probability of the event of **NO RED** beads. Then subtract from 1. We look at this below.*

5.5. How to Determine "At Least Once" Probabilities

87. It is near the end of the school term. The probability of passing the English test is **2/3** and the probability of passing the science test is **3/4** . What is the probability of passing **at least one** of the tests?

Let **E** and **S** represent the following events:

E: the event of passing the English test
S: the event of passing the science test

The probability of **failing** both tests is:

$$P(\bar{E}) * P(\bar{S}) = \frac{1}{3} * \frac{1}{4} = \frac{1}{12}$$

The opposite of complete failure is some success!

88. Therefore, the probability of passing *at least one* test is the **complemen**t of failing both tests. We can write this as:

$$P(Passing\ at\ least\ E\ OR\ S) = 1 - P(\bar{E}\ AND\ \bar{S}) = 1 - \frac{1}{12} = \frac{11}{12}$$

Therefore, the **probability of passing at least one test is 11/12**

89. An **"At Least Once"** probability is the complement of the probability that the events will never occur at all. To determine an "**at least once**" probability, we:

- first calculate the probability that the event does not occur.
- then find the complement of that probability.

90. We can write this as a formula:

P(the event occurs at least once) = 1 – P(the event does not occur)

or

$$\boldsymbol{P(E \geq once) = 1 - P(\bar{E})}$$

Think about it, say it out loud! The probability that something happens at least once is 1 minus the probability it never happens.

The number of times you lock yourself out of the house is at least the number of times you left the house and subtract the times you did not lock yourself out.

5.6. Definition of Dependent Events

91. Event A and event B are **dependent** events if the outcomes of event A influence the outcomes of event B. The probability that some dependent events will all occur is again the product of their separate probabilities but this time the probabilities may differ depending on how they affect each other. This may be written as the formula:

Here is what we will learn: $P(A\ then\ B) = \boldsymbol{P(A)} * \boldsymbol{P(B}|A)$;

Which is read as: '*the probability that event A happens then event B happens is the product of the probability of A happening with the probability of B happening given that A already happened*'.

The probability of me entering the room **and then** stubbing my toe on the corner of my desk is the probability that I actually entered the room times the probability that I stub my toe after entering the room.

Clearly if I never enter my room then I will not stub my toe on my desk!

Discuss.

For example, if you have a bag containing **six white** and **five black** marbles, and randomly **selected two** marbles, what is the **probability** that **both are black**? (The first marble is not returned to the bag before the second one is selected so it affects the next event's probability)

92. Let B_1 and B_2 represent the following events:

B_1: the first marble is black
B_2: the second marble is black

Then,

$$P(B1 \text{ and } B2) = P(B1) \bullet P(B2|B1)$$

or stated another way,

$$P(\text{both black}) = P(1^{st} \text{ black}) \bullet P(2^{nd} \text{ black} \mid 1^{st} \text{ black})$$

$P(B_2 \mid B_1)$ means "the probability of selecting a second black marble given that one black marble has already been taken on the first draw."

$$\therefore P(\text{both black}) = \frac{5}{11} \times \frac{4}{10} = \frac{2}{11}$$

93. After the first black marble has been taken, there are only four black marbles in the bag, and the total number of marbles in the bag is 10. For this reason:

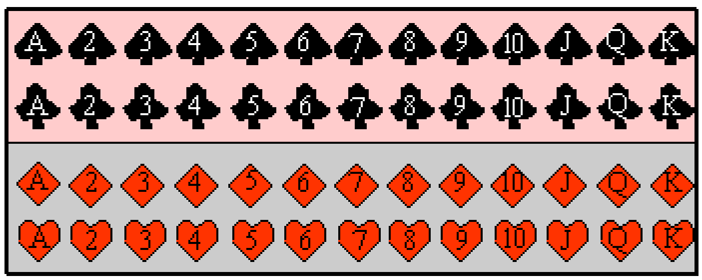
$$P(1^{st} \text{ black}) = \frac{5}{11}$$

$$P(B2|B1) = P(2^{nd} \text{ black} \mid 1^{st} \text{ black}) = \frac{4}{10}$$

$$\therefore P(\text{both black}) = \frac{5}{11} \times \frac{4}{10} = \frac{2}{11}$$

5.7. Example 1: Probability of Dependent Events

94. A deck of cards consists of the following cards:



95. What is the probability of drawing a red card and then a black 9 (without replacement of the first card)?

Prob (Red and then Black) = Prob(1^{st} Red) * Prob(Black 9 | 1^{st} draw was red)

$$= \frac{26}{52} * \frac{2}{51} = 0.0196 = 1.96\%$$

96. Given two draws, what is the probability of drawing a face card and then a 9.

Prob (**F and then 9**) = Prob (**F_1**) * Prob (**9_2 | F** drawn already)

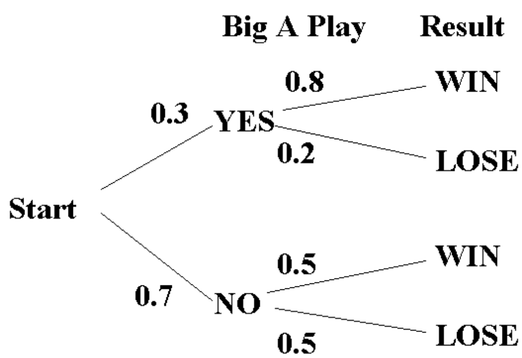
$$= \frac{12}{52} * \frac{4}{51} = 0.01810 = 1.81\%$$

97. If you want to confirm this is true, you could make an outcome tree! There will be 12 branches that have a Face Card drawn first, and each of those 12 branches will have 51 branches, 4 of which will have a 9. So there will be 48 outcomes that make the event Face Card then 9. But the total number of ways to draw two cards (without replacing is 52* 51 or 2652 ways). So the probability by counting the outcome is still $\frac{48}{2652} = 1.81\%$

5.8. Example 2: Probability of Dependent Events

98. The Moose have a game scheduled for December 30. If the star defence man **Big A** is able to play, the probability of winning is **0.8**, but if he does not play, the probability of winning is 0.5. The probability that Big A will play is 0.3 since he has a knee injury.

What is the probability the Moose will win on December 30?



Notice the difference from the independent probability tree. Can you see the difference?

99. There are two outcomes that have **WIN** for the 'Moose'. One is with Big A playing, the Prob(**WIN | BigA**) is 0.3 * 0.8 = 24%. Another favourable outcome is with Big A **not playing** but the 'Moose' still winning; Prob(**WIN | Not BigA Play**). The probability of that is 35%. So, the overall probability of the 'Moose' winning is **59%.**

5.9. Are the Events Independent or Dependent?

100. There were **350** students in **Math 101** at University. Of these 350 students, **60% had an 'A' in PreCalc 40S** (in high school) and **30% failed Math 101** in university. University Statistics show that **10% of the students had an A in PreCalc 40S** and **still failed Math 101**. Are getting an **A** in **Pre-Calc** and failing **Math 101** in university independent events? Explain your answer.

Solution:

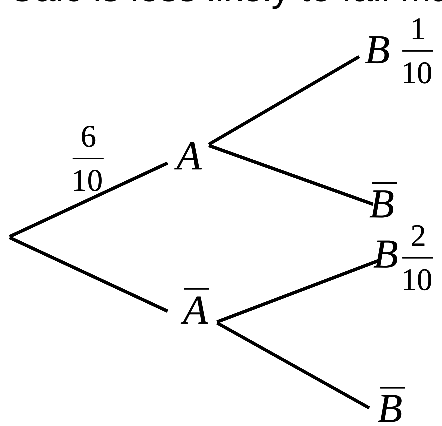
Let **A** and **B** represent the following events:

A: the event that a student got an A in Pre-Calc 40S
B: the event that a student fails Math 101

Then **P(A)** and **P(B)** are:

P(A) = 0.60
P(B) = 0.30, and
P(A and B) = 0.60 x 0.30 = 0.18

This indicates that if getting an A in Pre-Calc and failing Math 101 were independent, then the number of students that got an A in Pre-Calc and still failed should be 18% of the students. The statistics showed that only 10% of the students got an A in Pre-Calc and failed Math 101. Therefore, the two events are not independent. A student with an A in Pre-Calc is less likely to fail Math 101 at university.



5.10. Example 1: Are the Events Independent or Dependent?

101. Classify the following events as independent or dependent.

a) Choosing a red marble from one bag and a blue marble from another bag.

b) Getting two sixes when you roll two dice.

c) Randomly choosing seven digits for a password.

d) Drawing an ace for the first card and a king for the second card from a deck *if replacement is allowed*.

e) Drawing two cards separately if there is *no replacement*.

f) exceeding the speed limit and getting a speeding ticket.

g) winning the lottery and getting hit by a bus.

h) having a daughter born after three sons are born.

5.11. Lesson Summary

102. The following words (terms) or concepts were used in this lesson:

- "at least once" probability
- **complement** of an event
- dependent events
- independent events
- probability tree
- without replacement

You should know how to use the following formulas to solve problems covered in this lesson.

For **independent** events A and B,
P(A and B) = P(A) * P(B)

For **dependent** events A and B,
$P(A \text{ and } B) = P(A) \bullet P(B|A)$

To determine "at least once" probabilities,
P(the event occurs at least once) = 1 – P(the event does not occur)

- Notice that dependent events are normally characterized by one event happening before another.

Experiments with replacement and without replacement.

6. Lesson 6 - Mutually Exclusive and Non-Mutually Exclusive Events

6.1. Introduction

103. There are 26 students in Judy's class. Seven are in the school band, nine are members of sports teams, and three are in both the band *and* a sports team. What is the probability that a student selected at random is in the band *or* on a sports team?

In this lesson, we will study probabilities of events that may or may not have common outcomes.

6.2. Outcomes

When you have completed this lesson, you will be able to:

- identify mutually exclusive and non-mutually exclusive events
- use the addition rule of probability

6.3. Definition of Mutually Exclusive and Non-Mutually Exclusive Events

Two 6-sided dice are rolled. Consider events **E**, **F**, and **D** where:

E is rolling a sum of eight,
F is rolling a sum of four, and
D is rolling a double.

The sample space shows the outcomes when two dice are rolled.

1,1	1,2	1,3	1,4	1,5	1,6
2,1	2,2	2,3	2,4	2,5	2,6
3,1	3,2	3,3	3,4	3,5	3,6
4,1	4,2	4,3	4,4	4,5	4,6
5,1	5,2	5,3	5,4	5,5	5,6
6,1	6,2	6,3	6,4	6,5	6,6

The outcomes for event **E** are: (6,2) (5,3) (4,4) (3,5) and (2,6).
The outcomes for event **F** are: (3,1) (2,2) and (1,3).
The outcomes of event **D** are: (1,1) (2,2) (3,3) (4,4) (5,5) and (6,6).

You Circle events F and D above

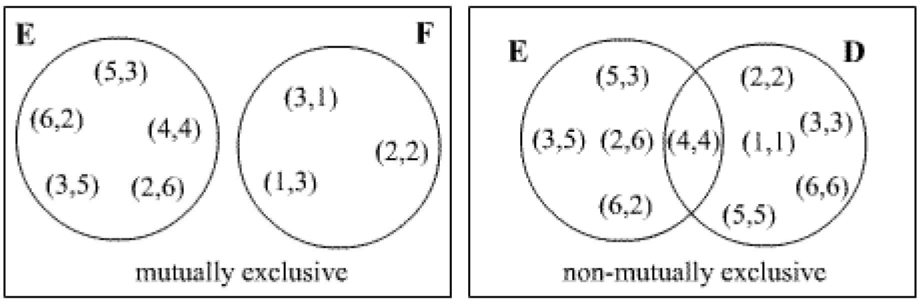
104. We say that events E and F are **mutually exclusive** **events** because they do not have any common outcomes. Two or more events whose outcomes can never be the same.

105. The outcomes of events E and D are **non-mutually exclusive events** because the outcome **(4, 4)** appears in both.

6.4. Diagrams of Mutually Exclusive and Non-Mutually Exclusive Events

106. The mutually exclusive events **E** and **F** from the previous page cannot occur at the same time. They have no common outcomes, as shown by the diagram on the left. below

107. The diagram on the right shows that the two events E and D are not mutually exclusive because they have a common outcome, which is (4,4).



108. These types of diagrams are called Venn diagrams. They show all the outcomes of an entire ***sample space*** *(all the simple pairs of numbers in this diagram)*. They also show those sets that represent certain ***events*** (the elements that are circled together in groups).

See the Annex to these Notes for a discussion of Venn Diagrams.

6.5. Which Events Are or Are Not Mutually Exclusive?

109. Decide whether the following are mutually exclusive or non-mutually exclusive events. A Venn diagram will help!

a. From a deck of cards: the event of drawing a heart and the event of drawing a 6.

b. From a deck of cards: the event of drawing a black card and the event of then drawing a diamond without replacing the first card?

c. Shoes that are runners and shoes that are black.

d. Persons in prison and persons innocent of a crime

e. People *under* 20 years of age (age<20) and people *over* 40 years of age (age>20).

f. Red poker cards and poker cards that are Spades

g. Being a cute person and a person having blue eyes

h. You invent one of each:
Mutually exclusive:

non-Mutually exclusive:

6.6. The "Addition Rule of Probability" [**OR**]

The information from the first page of the lesson is repeated here:

Two six-sided dice are rolled. Consider events E, F, and D where:

E is rolling a sum of eight,
F is rolling a sum of four, and
D is rolling a double.

The sample space shows the outcomes when two dice are rolled.

1,1	1,2	1,3	1,4	1,5	1,6
2,1	2,2	2,3	2,4	2,5	2,6
3,1	3,2	3,3	3,4	3,5	3,6
4,1	4,2	4,3	4,4	4,5	4,6
5,1	5,2	5,3	5,4	5,5	5,6
6,1	6,2	6,3	6,4	6,5	6,6

The outcomes for event E are: (6,2) (5,3) (4,4) (3,5) and (2,6).
The outcomes for event F are: (3,1) (2,2) and (1,3).
The outcomes of event D are: (1,1) (2,2) (3,3) (4,4) (5,5) and (6,6).

Formula for mutually exclusive events:

110. What is the probability of rolling a sum of 8 or 4 (i.e., P(E or F))? First, we determine the *number* of outcomes in events E and F.

n(E) = 5; n(F) = 3; n (Total) or n(S) for number in Sample space = 36
∴ n(E) + n(F) = 5 + 3 = 8

Now we determine the probability of event E

and P(E) = $\frac{n(E)+n(F)}{n(S)} = \frac{5+3}{36} = \frac{8}{36} = \frac{2}{9} = 22.22\%$

__Or__ almost always means __add__ in math problems; especially probability and logic

(**Note that the word *'or'* implies addition almost always in logic and probability when working with sets of numbers or events**.)

111. Therefore, the rule for finding the probability of mutually exclusive events **A** and **B** may be written as:

P(A or B) = P(A) + P(B) ***For Mutually Exclusive Events***

Formula for events that are *not mutually* exclusive:

112. What is the probability of the event rolling a **sum** of **8 or** a **Double** (i.e., **P(E ∪ D)**)? {Note we often use the symbol "∪" to mean '**OR**', it stands for the ***U****nion* of two sets if you have studied Logic and Set Theory}

First, we determine the number of outcomes in events **E** and **D**.

n(E) = 5; n(D) = 6; but events E and D have one outcome which is common: (4, 4), Since rolling a 4 and another 4 is both a sum of 8 and it is doubles.

∴ n(E) + n(D) – n(E AND D) = 5 + 6 – 1 = 10 ways

Now we determine the probability of event E OR D.

P(E ∪ D) = P(E) + P(D) – P (E ∩ D)

(where $\cap$ means the common i***n***tersection of **E** and **D**, the AND logic)

therefore **P(E $\cup$ D) =** $\frac{n(E)+n(D)-n(E\cap D)}{n(S)}$

$$= \frac{n(E)}{n(S)}+\frac{n(D)}{n(S)}-\frac{n(E\bigcap D)}{n(s)}$$

$$= P(E)+P(D)-P(E\bigcap D)$$

$$= \frac{5}{36}+\frac{6}{36}-\frac{1}{36}=\frac{5}{18}\cong 27.78\%$$

Therefore, the rule for finding the probability of 'non-mutually exclusive' events A and B may be written as:

P(A OR B) = P(A) + P(B) – P(A AND B); *often stated using logic and set notation as:*

P(A $\cup$ B) = P(A) + P(B) – P(A $\cap$ B)

7.7. Example 1: Mutually Exclusive Events

113. If one card is to be selected from a regular deck of 52 cards, find the number of ways in which a heart **or** a black ace can be drawn.

Solution

Notice the keyword: ‘**or’**

Let Event A = 'a heart is drawn' and Event B = 'a black ace is drawn.'

Let n(A) and n(B) are the numbers of ways in which events A and B can occur.

n(A) = 13, n(B) = 2. A and B are **mutually exclusive** events.

Then, n(A or B) = n(A) + n(B) =13 + 2 =15.

Also, P(A or B) = P(A) + P(B) = 13/52+2/52=15/52

114. For mutually exclusive events, P(A or B) = P(A) + P(B).

7.8. Example 2: Non-Mutually Exclusive Events

115. If one card is to be selected from a regular deck of 52 cards, find the number of ways in which a heart **or** an ace can be drawn.

Solution

Let Event A = 'a heart is drawn' and Event B = 'an ace is drawn'.

This time, **n(A)** = 13 and **n(B)** = 4. But, these events are not mutually exclusive because the ace of hearts belongs to both A and B.

n(A or B) = n(A) + n(B) – n(A and B)
n(A or B) =13 + 4 – 1 =16

P(A or B) =P(A) + P(B) – P(A and B) = 13/52+4/52-1/52=16/52=4/13

For non-mutually exclusive events, P(A and B) = P(A) + P(B) – P(A and B)

7.9. Example 3: Mutually Exclusive and Non-Mutually Exclusive Events

116. Thirty people go to a picnic in Manitoba in early July. Event A is (being bitten by at least one mosquito). Event B is (getting sunburned). Prob(A) =1/3. P(B) =1/5. It is. also known that P(being neither sunburned nor bitten by mosquitoes) = 1/2.

a) What is the least number of people who suffered both ways?

b) Are A and B mutually exclusive?

Solution:

Prob (A OR B) = Prob (A) + Prob (B) – Prob (A AND B)

We know **Prob (Not A)** and **Prob (not B) = 0.5**. Therefore, probability of either event A or B or both events is **0.5** (that is 1– 0.5). Or in other words Prob (**A OR B**) = 0.5.

So Prob (A and B) = Prob (A) + Prob (B) – Prob (A or B).

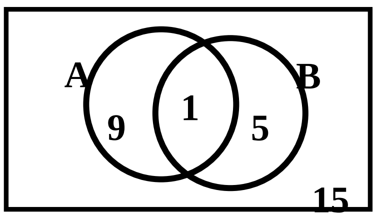
= 1/3 + 1/5 – ½

= 10/30 + 6 /30 – 15 / 30

= 1/30 So at least 1 out of 30 people has both mosquito bites and sunburn.

A and B are **not mutually exclusive**

Maybe this diagram from Set Theory which we study elsewhere will help:



7.10. Lesson Summary

The following words (terms) were used in this lesson:

- complement
- mutually exclusive
- not mutually exclusive
- Venn Diagram

You should know how to use the following formulas and concepts to solve problems covered in this lesson.

P(A or B) = P(A) + P(B) for **mutually exclusive** events A and B.

P(A $\cup$ B) = P(A) + P(B) for **mutually exclusive** events A and B.

P(A or B) = P(A) + P(B) – P(A and B) for **non-mutually exclusive** events.

P(A $\cup$ B) = P(A) + P(B) – P(A $\cap$ B) for **non-mutually exclusive** events.

Congratulations! You have reached the end of the unit and the course

MATH 40 APPLIED
PROBABILITY PERMUTAINS AND COMBINATIONS
GLOSSARY

Math 40 - Glossary

A	
Arrangement	The manner in which things are organized. Sometimes the order is important, sometimes it is not.

C	
Combination	The number of ways of selecting objects is a combination. (order is not important) Example: how many ways can 3 players on a team be selected for a league all-star game? It doesn't matter what order you get selected in, just that you make the team. The answer is ${}_{10}C_3$ or 120 ways *Compare to permutation*.
Complementary Events	If an event will definitely occur, then the probability of its occurrence is one. As a consequence, the probability of an event A *not* occurring, written **P(~A)**, is the difference between 1 and P(A) or **1 - P(A).** The complement of an event **A** is also shown sometimes as $\bar{A}$ **.** $\bar{A}$ and **~A** are read as**: 'Not A'**

D	
Dependent	Event A and event B are **dependent** events if the outcomes of event A influence the outcomes of event B
E	
Event	A set of outcomes, one or more outcomes of an experiment. A specific outcome or type of outcome. A subset of the sample space. **Example**: the outcome of drawing a single card is 52 possible outcomes, the event of drawing a **red** card is a set of 26 of those. The sample space is all equally likely 52 cards of the deck, the subset is the event (RED CARD)

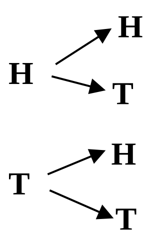
Expected value	An estimate of the average expected return or loss you will have. Expected Value (**EV**) = P(win) * (Gain) – P(lose)* (Loss) **Example**: If you have a probability of 20% of Winning $10.00 and for an investment of $4.00. EV = 0.2*(10 – 4) – 0.8 * (4) = 1.2 – 3.2= ⁻2.0 If the EV is <0 you can expect to lose that amount every play; If the EV = 0 you will break even; and If the EV > 0 you can expect to gain that amount every play. So in the above game you can expect to lose. If you play 100 times you can expect to lose 2.0*100 = $200.00
Experimental Probability	The chance of an event happening based upon repeated testing or observed trials.

F	
Factorial Notation	In general, n factorial is **n! = (n)(n – 1)(n – 2) ... (3)(2)(1).** **Note that 0! ≡ 1**
Fundamental Counting Principal	Suppose that an event **K** can occur in ***k*** number of ways and after it has occurred, event **M** can occur in ***m*** ways. Thus, the number of ways in which both **K** and **M** can occur is ***k*m*** ways.

I	
Independent Events	Events A and B are independent if the probability of A is not influenced by the probability of B. Two events A and B are independent if P(A) = P(A | B) or P(B) = P(B | A).

M	
Mutually Exclusive Events	When two events cannot occur at the same time because they do not have any common outcomes these events are said to be mutually exclusive. **Example**: The events of drawing a red card and the event of drawing a spade, are mutually exclusive. They are mutually exclusive because there are no red spades ♠.

O

Outcome	A possible result of a single trial of an experiment, a possible answer to a survey question. EG: for the experiment of tossing a six-sided dice the possible outcomes are 1,2,3,4,5,6. The entire collection of outcomes possible forms the sample space
Odds Against	Ratio of unfavourable outcomes to favourable outcomes. **Example**: The **probability** of rolling a sum of 12 with 2 dice is 1/36. Therefore the odds ***against*** rolling a 12 are **35:1**. Because there are 35 ways to **not** roll a 12 and only 1 way to roll it. Normally odds are usually expressed as *odds in favour* though.
Odds in Favour	ratio of favourable outcomes to unfavourable outcomes Note that a probability of drawing a Heart from a deck of cards is 13/52 = ¼. The odds in favour are therefore **13: 39** or **1:3**
Outcome Tree	A tree showing all possible outcomes. This can get to be very big. **Example**: All possible outcomes of flipping two coins:  There are four possible outcomes. Only one is Tails, Tails. Therefore probability of rolling TT is 1 in 4 or 0.25 or 25%

P

Permutation

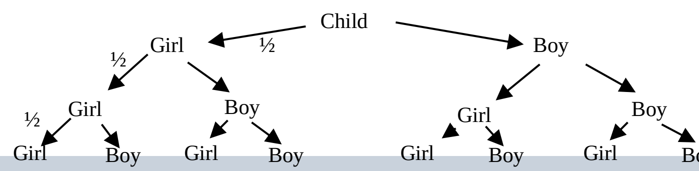
The number of ways of **selecting** and **ordering** objects is a permutation (two actions).
Example: how many ways from 10 players from a team be ***selected in order*** for a first, second, and third prize? Answer is $_{10}P_3 = 720$

Probability

probability of the occurrence of Event A is the ratio of the number of occurrences of A to the total number of possible outcomes. Probability of an event is always 0 to 1 in decimal or 0 to 100% in percent.

Probability Tree

A tree showing all possible outcomes by probability. Usually better for doing probabilities. **Example** probability of having 3 daughters. The probability is ½ * ½ * ½=1/8 = 12.5%



R

Random

A random experiment is an experiment which can result in different outcomes, and for which the outcome is unknown in advance. There is no bias in the experiment.

S

Sample Space

A sample space is the set of all possible outcomes of a trial or survey.

Examples:

- The sample space of rolling a normal six-sided dice is the outcomes: {**1, 2, 3, 4, 5, 6**}.
- The sample space of tossing a single coin is the outcomes {**H, T**}.
- The sample space of genders of three children in a family is {**GGG, GGB, GBG, GBB, BGG, BGB, BBG, BBB** }

Simulation

A mathematical experiment that approximates a real-world process. Example: You can simulate how many of each gender of child you will have by using a coin. Head is boy, tail is girl for example.

T

Theoretical Probability

The chance of an event happening as determined by the mathematical result that it will occur.

Tree

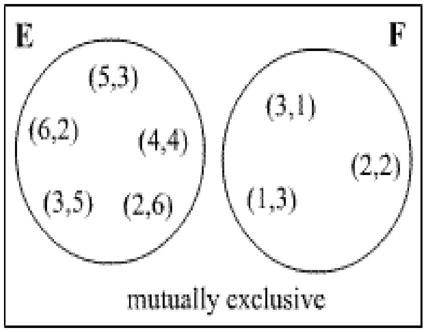
Two types. An outcome tree to count all the possible items in a sample space or a probability tree when paths have different probabilities.

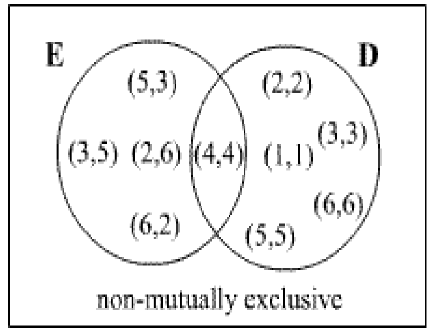
Trial

A sample; one of many tests to get a result in an experiment. A trial has an outcome.

V

Venn Diagram





A diagram that uses circles to show relationships among sets of numbers or objects.

A Table of the *Permutation Function* $_nP_r$

nPr	r										
n	0	1	2	3	4	5	6	7	8	9	10
1	1	1									
2	1	2	2								
3	1	3	6	6							
4	1	4	12	24	24						
5	1	5	20	60	120	120					
6	1	6	30	120	360	720	720				
7	1	7	42	210	840	2,520	5,040	5,040			
8	1	8	56	336	1,680	6,720	20,160	40,320	40,320		
9	1	9	72	504	3,024	15,120	60,480	181,440	362,880	362,880	
10	1	10	90	720	5,040	30,240	151,200	604,800	1,814,400	3,628,800	3,628,800

Example Use:

How many ways can you select 5 different people, **r**, from a total group, **n** , of 5 people on a bench. $_5P_5 = 120$

How many ways can I stack my 6 of my 8 favourite math books on my desk: $_8P_5 = 6{,}720$ **ways**

How many ways can 10 runners take 1st, 2nd, and 3rd place in a race. $_{10}P_3 = 720$

If I have 9 different coupons to hand out to students, how many ways can I hand out none of them? $_9P_0 = 1$

ORDER MATTERS with Permutations

Combinations Formula – Table. $_nC_r$

nCr n \ r	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	1														
2	1	2	1													
3	1	3	3	1												
4	1	4	6	4	1											
5	1	5	10	10	5	1										
6	1	6	15	20	15	6	1									
7	1	7	21	35	35	21	7	1								
8	1	8	28	56	70	56	28	8	1							
9	1	9	36	84	126	126	84	36	9	1						
10	1	10	45	120	210	252	210	120	45	10	1					
11	1	11	55	165	330	462	462	330	165	55	11	1				
12	1	12	66	220	495	792	924	792	495	220	66	12	1			
13	1	13	78	286	715	1,287	1,716	1,716	1,287	715	286	78	13	1		
14	1	14	91	364	1,001	2,002	3,003	3,432	3,003	2,002	1,001	364	91	14	1	
15	1	15	105	455	1,365	3,003	5,005	6,435	6,435	5,005	3,003	1,365	455	105	15	1

Example Usage:

How many ways can a committee of 4 people be selected from 12 people? **n = 12, r = 4.** $\mathbf{{}_{12}C_4 = 495}$

If I have to submit 6 student tests out of 10 to the government for them to mark how many possible ways could I have to possibly bundle up 6 out of 10 tests? $\mathbf{{}_{10}C_6 = 210}$ to **bundle** them up! (I hope it is my top 6 students that I have to send in). A bundle has no order!

Do you notice how this table relates to path way problems?? Notice how rows sum up to are $\mathbf{2^n}$.
If I have to make 9 steps and three have to be up, the rest right, how many different paths can I take? $\mathbf{{}_9C_3}$ or $\mathbf{{}_9C_6 = 84}$; the same thing!

ORDER DOESN'T MATTER with Combinations. Like the Lotto 6/49!

Notice how $_6C_2$ is the same as $_6C_4$. Notice how $_{10}C_1$ is the same a $_{10}C_9$. Notice how $_3C_2$ is the same as $_3C_1$. Notice how 9C0 is the same as $_9C_9$.

See a pattern? $_nC_r$ is the same as $_nC_{(n-r)}$. Knowing which objects are not chosen is the same as knowing which ones are!

FACE CARDS

Example set of 52 poker playing cards

Suit	Ace	2	3	4	5	6	7	8	9	10	Jack	Queen	King
Clubs													
Diamonds													
Hearts													
Spades